TIDAL HYDRAULICS

BY

BRIGADIER GENERAL GEORGE B. PILLSBURY
U. S. A., Retired

NOVEMBER 1939

WAR DEPARTMENT
CORPS OF ENGINEERS, U. S. ARMY

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PREFACE

The treatment of the tides, and of tidal datum planes, contained in the first four chapters of this text, and of the reduction of measured tidal currents, in chapter X, is drawn from the manuals issued by the United States Coast and Geodetic Survey, and from Harris's Manual of Tides, published in past reports of that Survey, but now out of print. As no engineer outside that Survey may expect the occasion to undertake the laborious harmonic analysis of the tides at a station, the voluminous tables required for the purpose are not included.

The cubature of a channel, described in chapter VI, is set forth in a number of French texts. The detailed procedure explained is that developed in the United States Engineer office at Philadelphia.

A method is developed in chapters V and VIII for computing tidal currents from the constants commonly used for steady flow, by a procedure somewhat analogous to that used in ordinary hydraulic computations. Quite obviously, the varying and periodically reversing flow in a tidal channel has somewhat the same relation to steady flow that an alternating electric current has to a direct current. As alternating currents depend upon the reactance and capacity of the circuit as well as upon its resistance, so tidal currents depend upon the acceleration head and the storage and release of water in the channel as well as upon frictional resistance. When these factors are included, computations of tidal flow should be as reliable as are those for steady flow. The application of these principles to natural tidal channels is taken up in chapter IX.

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Tides

THE TIDES

Chapter I

GENERAL DEFINITIONS: THE TIDE-PRODUCING FORCES: EQUILIBRIUM TIDES

GENERAL DEFINITIONS

1. The *tide* is the regular periodic rise and fall of the surface of the seas, observable along their shores. The concurrent horizontal movements of the water, whether the almost imperceptible drift in the open sea, or the strong flow through a contracted entrance to a tidal basin, are designated, in accordance with the practise of the United States Coast and Geodetic Survey, as *tidal currents*.

2. *High and low water.*—The maximum height reached by each rising tide is called *high water*, and the maximum depression of the falling tide is called *low water*. On the Atlantic coast of the United States the tide rises and falls twice daily—or more accurately twice during the lunar day of 24 hours and 50 minutes. The two high waters and the two low waters are each so nearly equal that for ordinary purposes no distinction need be made between them. On the Pacific coast the two high waters and the two low waters occurring daily are in general markedly different, and are designated as the *higher high water*, the *lower high water*, the *lower low water*, and the *higher low water*. On the Gulf of Mexico the tides are small, and toward its western end but one tide occurs each day during a part of the month.

The heights of the high waters and of the low waters vary from day to day. In many parts of the world, the high waters reach their greatest height, and the low waters the least height, soon after the

time of full and new moon. These tides are called *spring* tides. The term "spring" as applied to tides has nothing to do with the season of the year, but is the greater upspringing of the waters at intervals of about a fortnight. Similarly the daily high waters are usually at their least height, and the daily low waters their greatest height, soon after the moon is in quadrature. These tides are called *neap* tides. On the Atlantic coast of Europe and along the British Isles the difference between low or high water of spring tides and low or high water of neap tides may amount to several feet, and is a matter of moment to navigators. On the coasts of the United States the difference between spring and neap tides is not particularly noticeable, and the terms "spring" and "neap" tides are not in ordinary use. In this country spring tides are commonly referred to as "tides at full (or new) moon" or occasionally as "moon tides."

3. *Datum planes.*—The average height of all low waters at any place over a sufficiently extended period of time is called *mean low water* and is the official reference plane for the depths shown on navigation charts, and of improved channels, in the waters of the Atlantic and Gulf coasts of the United States. The average height of the lower of the two daily low waters is called *mean lower low water* and is the official reference datum in the waters of the Pacific coast of the United States. In British waters the datum is usually the mean low water of spring tides, or *low-water springs*. This reference plane is also used at the Pacific entrance to the Panama Canal. The average height of the sea, as determined usually by the average of the observed *hourly* heights over an extended period of time, is called *mean sea level*, and is the standard datum to which elevations on land are referred.

4. *Tidal ranges.*—The difference in height between high water and low water at a tidal station is called the *tidal range*. The *mean range* is the average of the differences between all high waters and all low waters; or, as is the same thing, the difference between *mean high water* and *mean low water* at the station. The *diurnal range*, or *great diurnal range*, is the difference between mean higher high water and mean lower low water. The *extreme range* is the maximum that has been observed. The *spring range* is the difference between mean high water and mean low water of spring tides, and the *neap range* the difference between mean high water and mean low water of neap tides.

5. *Tidal currents.*—The tidal current setting into the bays and estuaries along the coast is called the *flood current*. The return current toward the sea is called the *ebb* current. The maximum velocities reached during each fluctuation of the current are called the *strength of the flood* and the *strength of the ebb*, or, indifferently, the *strength of the current*. *Slack water* is the period during which the current is

negligible while it is changing direction. It is specifically defined by the United States Coast and Geodetic Survey as the period during which the current is less than one-tenth of a knot; i.e., less than 0.169 feet per second. The slack water occurring nearest the time of high water is called the *high-water slack*, and that nearest the time of low water the *low-water slack*. The moment at which the current is zero as it changes direction may be distinguished by terming it the *turn of the current*.

In open waters, the direction of the current normally veers around the compass and the current does not pass through intervals of slack water. Such currents are called *rotary*, to distinguish them from the *reversing* currents in a tidal channel.

6. These definitions are narrower than the common usage of the terms. "Tide" is commonly applied both to the rise and fall of the sea and to the accompanying tidal currents. Thus the expressions "head tide" and "favoring tide" designate tidal currents that retard or accelerate the movement of a vessel, and the term "the ebb and flow of the tide" is standard legal nomenclature. The term "ebb tide" is often used to designate *low water* as well as the outflowing tidal current. The maximum tidal stage is frequently designated as "high tide" instead of "high water." Its more general meaning is, however, the higher stages of the tide. Thus it is more accurate to say that a channel is "navigable only at high tide," than to say that it is "navigable only at high water."

7. *Lunitidal intervals.*—Casual observation shows that the tides at any place occur a little less than 1 hour later each succeeding day. Thus if high water is at 3 p. m. today, it will be shortly before 4 p. m. tomorrow. Closer observation shows that the high and low waters at any place follow, by about the same time interval, the passage of the moon across the meridian of the place. Obviously, the moon must cross the plane of the meridian twice daily—once overhead and once underneath. These are called respectively the *upper* and *lower meridian transits*. They mark in fact the noon and midnight of the lunar day. If a clock were regulated on mean lunar time, instead of mean solar time, it would show the times the high and low waters at a given place at about the same hour every day, but these times would vary largely from place to place.

8. The *average* time interval, in solar hours and minutes, from a lunar transit to the next succeeding high water at a given place, as determined by an extended set of observations, is called the *high-water interval*, (HWI) or the *high-water lunitidal interval* of the place. Similarly the *low-water interval* (LWI), or the *low-water lunitidal interval* is the average time, in solar hours and minutes, from a lunar transit to the next succeeding low water. The high- and low-water intervals usually are larger at the full and change of the moon, at about

the time of spring tides, than at other times in the month. Charts of foreign waters sometimes give the intervals at such times, instead of the mean intervals, designating them as HWI, F. & C., and LWI, F. & C., respectively.

9. The charts of the United States Coast and Geodetic Survey and other publications show the average lunitidal intervals at representative tidal stations. By computing from the Nautical Almanac the times of upper and lower meridian transits of the moon at the place on any day, the times of high water on that day can be approximately determined. Although rarely of practical importance, the method of computation is of interest.

The Nautical Almanac gives the Greenwich mean solar time of the moon's upper and lower transits across the meridian of Greenwich for each day in the year. This time obviously is the interval, or hour angle, between passage of the (mean) sun and the passage of the moon over the Greenwich meridian. This interval increases at the average rate of 25.2 minutes every 12 hours, or 2.1 minutes per hour. If then the longitude of a given place is $L°$ west of Greenwich, the transit of the (mean) sun over its meridian will be $L°/15$ hours later than the transit over the Greenwich meridian, and the interval between the transits of the (mean) sun and of the moon over the meridian of the place, or the local mean solar time of the moon's transit, will be the Greenwich time of transit increased by 2.1 $L°/15$ minutes. For example, the high-water interval at Sandy Hook, long. 74° W., at the entrance to New York Harbor, is $7^h.35^m$. For April 12, 1936, the Almanac gives:

	Upper	*Lower*
Moon's transit, Greenwich	3^h55^m	16^h21^m
Correction to Sandy Hook (74/15) 2.1	10^m	10^m
Local time moon's transit, Sandy Hook	4^h05^m	16^h31^m
Correction to standard time 75° meridian	-04^m	-04^m
Standard time moon's transit, Sandy Hook	4^h01^m	16^h27^m

Adding the high-water interval to the times of the moon's transits, the approximate times of high water at Sandy Hook are found to be 11^h36^m and 24^h02^m; or 11:36 a. m. April 12 and 12:02 a. m. on April 13. The times given in the tide tables are 11 a. m. and 11:24 p. m. on April 12. The time of high and low water found from lunitidal intervals may be in error by half an hour or more.

10. The difference between the lunitidal intervals at two tidal stations, corrected if necessary for the difference in the longitudes of the stations, gives the average difference between the times of high, or low, water at these stations. The formula for this correction is at once derived from the process of finding the time of high (or low) water from the Greenwich meridian transit of the moon and lunitidal interval, as set forth in paragraph 9. Let G be the time, in hours, of

a Greenwich lunar transit, I_1 and I_2 the lunitidal intervals at the two stations, L_1 and L_2 their longitudes in degrees west of Greenwich, S the longitude of the standard time meridian of the locality, and T_1 and T_2 the standard time, in hours, of high (or low) water at the two stations. Then:

$$T_1 = G + (2.1/60)\,(L_1/15) + (L_1 - S)/15 + I_1$$
$$= G + (4.14/60) L_1 - S/15 + I_1$$
$$T_2 = G + (4.14/60) L_2 - S/15 + I_2$$

Whence:

$$T_1 - T_2 = I_1 - I_2 + (4.14/60)\,(L_1 - L_2) \tag{1}$$

The correction for longitude is therefore 4.14 minutes of time for each degree of difference between the longitudes of the two stations, due regard being had to the algebraic sign of the correction resulting from the application of the formula. Obviously for easterly longitudes the sign of the correction would be reversed.

For example, the high-water interval at Portland, Oreg., long. 122°40′ W., is 6^h43^m, and at Astoria, near the mouth of the Columbia River, long. 123°46′ W., the high-water interval is 0^h41^m. The difference in the time of high water between Portland and Astoria is therefore $6^h43^m - 0^h41^m + 4.14\ (122.56 - 123.77)^m = 6^h02^m - 05^m = 5^h57^m$. High water at Portland is therefore 5^h57^m later, on the average, than high water at Astoria.

11. Since the time of high water cannot be determined from observation within a range of several minutes, the correction for the difference in longitude between two stations may be neglected unless it exceeds 1 minute of time. The corresponding difference in longitude is about 15′ of arc. No correction for longitude need be made therefore unless the two stations are at least 10 miles apart in an east and west direction.

12. *Greenwich lunitidal intervals.*—A Greenwich high- (or low) water interval at a station is the interval from a transit of the moon over the meridian at Greenwich, as given in the Nautical Almanac, to the Greenwich time of the following high (or low) water at the station. For convenience, high- and low-water intervals usually are computed by subtracting the tabulated Greenwich times of upper or lower transits from the time of the next ensuing observed high and low waters, as recorded on standard time at the station. The average differences so found are then converted to Greenwich intervals by adding the west longitude, in hours, of the standard-time meridian. If the result exceeds the average interval of 12.42 hours between successive lunar transits, that interval is subtracted. The local lunitidal intervals may then be found by subtracting the product of the west longitude of the station, in degrees, times 0.069 hours (4.14

minutes), from the Greenwich intervals, increased if necessary by 12.42 hours.

Thus, the mean interval from the tabulated Greenwich transits to standard times of observed high water at Seattle, Wash., in January 1928 was found to be 4.98 hours. As the standard time meridian at the locality is 120° west of Greenwich, the Greenwich interval is found by adding $120/15=8$ hours. As the sum, 12.98 hours, exceeds the interval between lunar transits, the average Greenwich interval at the station during the month is recorded as $12.98-12.42=0.56$ hours. The longitude of the station is 122°20′ W. The correction to be subtracted from the Greenwich interval to give the local lunitidal interval is $(122\frac{1}{3})\times 0.069=8.44$ hours. The average local high-water interval for the month is then $12.98-8.44=4.54$ hours.

While lunitidal intervals are conventionally given as local intervals, the Greenwich intervals are more convenient for most purposes, since the difference between the times of high (or low) water at any two stations is given directly by the differences in their Greenwich intervals, without correction for the different longitudes of the stations.

13. *Establishment of the port.*—The high-water interval at the full and change of the moon is called, in England, the "establishment of the port," and the high-water interval at spring tides the "corrected establishment." These terms are not current in the United States.

While the time of full moon is commonly thought of as a day, it is in fact an instant, duly set forth in the Nautical Almanac. The moon's transit nearest the moment of full or change evidently is nearly but not quite at noon or midnight, and the mean solar time of high water is close to the high-water interval. The establishment of the port is also defined therefore as the local time of high water at the full and change of the moon. The term is not further used in the treatment of the tides herein followed.

THE TIDE-PRODUCING FORCES

14. It is an elementary principle of physics that the gravitational attraction between two bodies varies inversely as the square of the distance separating them; and an elementary theorem that the attraction between two spheres, such as the moon and the earth, is the same as though their respective masses were concentrated at their centers. But the attraction between the moon and any individual unit of mass in the earth depends upon the distance of this unit from the center of the moon, which is not, in general, the same as the distance from the earth's center to the center of the moon. The consequent varying differential in the force of attraction over the earth's surface as compared with the average attraction per unit of mass of

the earth as a whole, together with a similar differential with respect to the attraction of the sun, are the tide-producing forces.

15. *The tide-producing force of the moon.*—In figure 1, C is the center of the earth, O the center of the moon, and P any point at or within the earth's surface, r the distance CP, a the radius of the earth, R the distance between the center of the earth and the center of the moon, D the distance from P to the center of the moon, θ (theta) the angle between CO and CP, and P the angle between PO and CP produced.

Let M be the mass of the moon,

μ (mu) the gravitational attraction between two units of mass at one unit's distance.

The attraction of the moon on a unit of mass at the point P is then $M\mu/D^2$ acting in the direction of PO, and its component in the direction CP is $(M\mu/D^2)\cos P$. Similarly the attraction of the moon on a unit of mass at the center of the earth is $M\mu/R^2$ acting in the direction CO, and its component in the direction CP is $(M\mu/R^2)\cos\theta$. The component of the difference of these forces, in the direction CP, is:

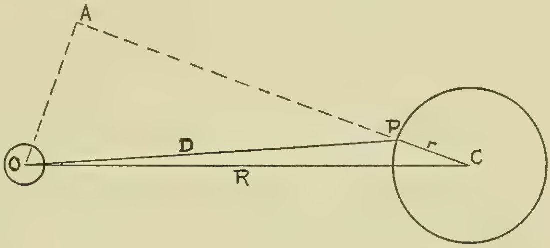

FIGURE 1.

$$fr = (M\mu/D^2)\cos P - (M\mu/R^2)\cos\theta \tag{2}$$

Let A be the foot of a perpendicular from the center of the moon, O, to the line CP produced. Then:

$$D\cos P = PA. \quad R\cos\theta = CA = PA + r.$$

whence:

$$R\cos\theta = D\cos P + r. \quad \cos P = (R\cos\theta - r)/D.$$

Giving:

$$\begin{aligned} fr &= M\mu\,[(R\cos\theta - r)/D^3 - \cos\theta/R^2] \\ &= M\mu\,[(\cos\theta - r/R)\,(R^3/D^3) - \cos\theta]/R^2. \end{aligned} \tag{3}$$

From the triangle POC:

$$D^2 = R^2 + r^2 - 2Rr\cos\theta.$$

Whence:

$$D^2/R^2 = 1 - 2(r/R)\cos\theta + (r/R)^2.$$

Placing, for convenience, $r/R=p$:

$$R^3/D^3=(1-2p\cos\theta+p^2)^{-3/2}$$
$$=[1-p(2\cos\theta-p)]^{-3/2}.$$

Expanding the second member into the binomial series:

$$R^3/D^3=1+3/2\,p(2\cos\theta-p)+15/8\,p^2(2\cos\theta-p)^2+\cdots$$
$$=1+3p\cos\theta-3/2\,p^2(1-5\cos^2\theta)+\text{ terms in the cubes and higher powers of } p.$$

Since the distance from the moon to the earth is approximately 60 times the earth's radius, the cubes and higher powers of $p=r/R$ have values of 1/216,000 or less, and the terms containing them are too small to be considered. Substituting, in equation (3), the expression derived for R^3/D^3, reducing and again dropping the cubes of p:

$$fr=M\mu\,[p(3\cos^2\theta-1)+3/2\,p^2(5\cos^3\theta-3\cos\theta)]/R^2$$
$$=M\mu\,(r/R^3)\,(3\cos^2\theta-1)+3/2\,M\mu(r^2/R^4)\,(5\cos^3\theta-3\cos\theta). \quad (4)$$

The numerical value of the coefficient of the second term of equation (4) is $3r/2R$ times, or in the order of 1/40th or less of, the numerical value of the coefficient of the first term. For the accuracy in general necessary, the second term may be disregarded, giving:

$$fr=M\mu(r/R^3)\,(3\cos^2\theta-1). \quad (5)$$

The distance of the moon from the earth is astronomically measured by its *parallax*, which may be defined as the angle subtended by the radius of the earth at the distance of the moon. The parallax varies as the reciprocal of the distance, or as $1/R$. Since the second term of equation (4) contains $1/R$ to the fourth power, it is called *the term dependent on the fourth power of the moon's parallax.*

16. The component of the lunar differential attraction in the direction perpendicular to CP, in the plane CPO, is similarly:

$$fh=(M\mu/D^2)\sin P-(M\mu/R^2)\sin\theta$$

from figure 1:

$$D\sin P=OA=R\sin\theta,$$

giving:

$$\sin P=R\sin\theta/D,$$

so that:

$$fh=M\mu(R\sin\theta/D^3-\sin\theta/R^2)$$
$$=M\mu\sin\theta(R^3/D^3-1)/R^2.$$

Substituting the expression for R^3/D^3 previously found, but dropping the *squares* and higher powers of p:

$$fh = M\mu \sin\theta(1+3p\cos\theta-1)/R^2$$
$$=3M\mu(r/R^3)\sin\theta\cos\theta=3/2\, M\mu(r/R^3)\sin 2\theta. \qquad (6)$$

The terms containing "the fourth power of the moon's parallax" being omitted.

17. When P is at the surface of the earth, r becomes a, the earth's radius. The line CP is evidently the vertical at P. Therefore the vertical component of the lunar tide-producing force is:

$$fr = M\mu(a/R^3)\ (3\cos^2\theta-1) \qquad (7)$$

and the horizontal component, in the direction of the moon, is:

$$fh = 3/2\, M\mu(a/R^3)\sin 2\theta. \qquad (8)$$

Since the vertical line CP is directed toward the zenith of the place P, it is also clear that the angle θ is the zenith distance of the moon, or the complement of the moon's altitude above the horizon.

18. *Characteristics of the lunar tide-producing force.*—It is evident from equation (7) that the vertical component of the tide-producing

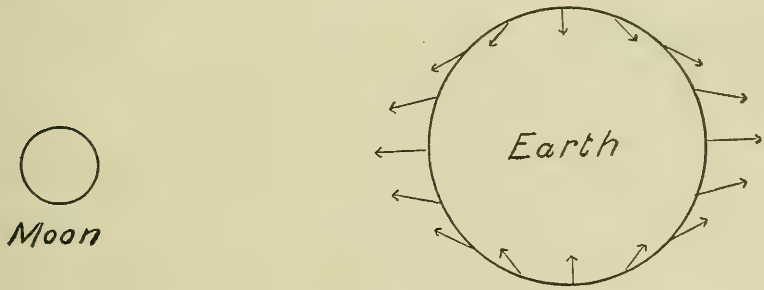

FIGURE 2.—Directions of tide-producing force.

force is a maximum when $\theta=0$ and $180°$ and is then $2M\mu a/R^3$. It is zero when $\cos\theta=\sqrt{1/3}$; i. e., when θ is 54°44′, 125°16′, 234°44′, and 305°16′. It reaches a maximum negative value of $-M\mu a/R^3$ when $\theta=90°$ and $270°$. Similarly the horizontal component increases from zero, when $\theta=0$, to a maximum of $3/2\ M\mu a/R^3$ when $\theta=45°$, and then decreases to zero when $\theta=90°$, repeating this variation with appropriate changes in sign in each quadrant. The resultants of the horizontal and vertical components of the tide producing force, for various values of θ, are shown graphically in figure 2.

The attraction of the moon tends to pull the water of the oceans toward it on the side of the earth nearest the moon, and to pull the

earth away from the water on the other side. The resultant tide-producing forces on the side of the earth away from the moon must balance the tide-producing forces on the side toward the moon, for otherwise the total attraction between the earth and the moon would not be the same as though the respective masses of these two bodies were concentrated at their centers. Because, however, of the somewhat greater attraction by the moon on the nearer area of the earth, the tide-producing forces on the two sides of the earth are not exactly symmetrical. This variation in the tide-producing force is expressed by the term containing the fourth power of the moon's parallax. It tends to mould the surfaces of the ocean into a very slightly pear-shaped variation from a perfect oval (fig. 8, par. 32).

19. *The solar tide-producing force.*—Designating the mass of the sun by S, and its distance from the earth by R_1, and its zenith distance at the point P by θ_1, the vertical component of the solar tide-producing force at the earth's surface is evidently, from equation (7):

$$fr_1 = S\mu(a/R_1^3)\ (3 \cos^2 \theta_1 - 1) \tag{9}$$

and the horizontal component, from equation (8):

$$fh_1 = 3/2 S\mu(a/R_1^3) \sin 2\theta_1 \tag{10}$$

The maximum value of the vertical component is $2S\mu a/R_1^3$. Its ratio to the maximum value of the vertical lunar component is:

$$(2S\mu a/R_1^3)/(2M\mu a/R^3) = (S/M)(R^3/R_1^3)$$

The mass of the sun, S, is 27,000,000 times the mass of the moon, M; but the distance of the sun from the earth, R_1, is about 389 times the distance, R, of the moon from the earth. Substituting these values, the ratio of the maximum values of the solar to the lunar tide-producing force becomes 27,000,000/58,863,869=0.46. Despite its enormously greater mass, the tide-producing force of the sun is less than half that of the moon, because of its greater distance.

20. A consideration of figure 3 shows that when the moon is full, M'', or at change, M', the solar tide-producing force will tend to increase the lunar tide-producing force, while when the moon is at quadrature, at M''' and M^{iv} the solar tide-producing force will tend to decrease the lunar tide-producing force. At the full and change of the moon, therefore, high waters tend to be higher and low waters lower, than at other phases of the moon, thus producing the spring tides at full and change, and neap tides at quadrature (par. 2).

21. *The tide-producing forces are minute.*—The force of gravity at every point on the earth's surface is $E\mu/a^2$, E being the mass of the earth, a its radius, and μ the gravitational attraction between two

units of mass at one unit of distance. The ratio of the vertical component of the lunar tide-producing force to the force of gravity is, from equation (7):

$$(M\mu a/R^3)(3 \cos^2 \theta-1)/(E\mu/a^2)=(M/E)(a^3/R^3)(3 \cos^2 \theta-1)$$

This ratio reaches a maximum of $2(M/E)(a^3/R^3)$ when $\theta=0$.

Since the mass of the earth is approximately 80 times the mass of the moon, and its distance from the moon approximately 60 times the earth's radius:

$$M/E=1/80 \qquad a/R=1/60$$

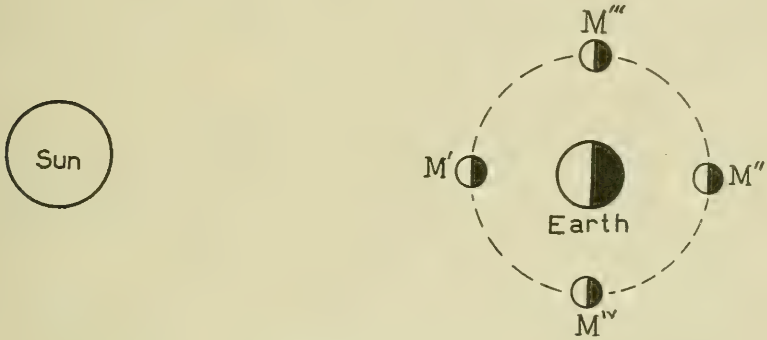

FIGURE 3.—Phases of moon.

The substitution of these values shows that the maximum value of the vertical component of the lunar tide-producing force is about 1/8,640,000 of the force of gravity. The maximum value of the horizontal component is similarly found to be about 1/17,280,000 of the force of gravity. The maximum values of the components of the solar tide-producing force are less than half of those of the lunar components. Such small forces evidently are not directly measurable by the most delicate instruments, nor can they sensibly affect the levels of limited bodies of water even as large as the Great Lakes. The accumulated effect of these small forces over the vast areas of the oceans is however sufficient to produce the tides.

THE TIDE-PRODUCING POTENTIAL

22. The effect of the tide-producing forces upon the waters of the oceans is indicated by the *potentials* of these forces. The *potential* of a force at any point is defined as the work required to move a unit of mass against the force to a position where the force is zero. Since the tide producing force is zero at the earth's center, the tide-producing potential at P, distant r from the center C (fig. 4) is the

work required to move a unit of mass against the force, from P to C. If the mass be moved along the radius PC, the radial component is the only part of the force against which work is done. The radial

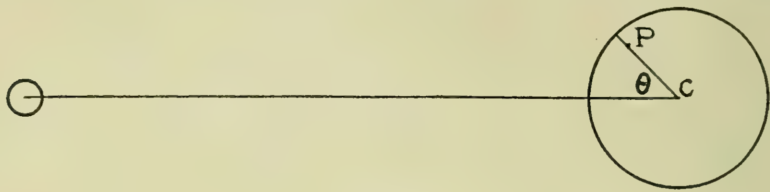

FIGURE 4.

component of the lunar tide-producing force is as shown in equation (5):

$$M\mu(r/R^3)(3\cos^2\theta-1)$$

As derived, this force is positive in the direction CP.

The lunar tide-producing potential at P is therefore:

$$V_t=-\int_r^0 M\mu(r/R^3)(3\cos^2\theta-1)dr=-(M\mu/R^3)(3\cos^2\theta-1)\int_r^0 rdr$$
$$=\tfrac{1}{2}M\mu(r^2/R^3)(3\cos^2\theta-1) \qquad (11)$$

23. *Relation of potential to force.*—It follows from the definition of the potential of a force, that its rate of change, in any direction, is the component of the force acting in that direction. Thus the rate of change of the lunar tide-producing potential in a direction perpendicular to the radius (in the plane of the moon, the point, and the earth's center) is:

$$dV_t/d(r\theta)=dV_t/rd\theta=\tfrac{1}{2}M\mu(r/R^3)d(3\cos^2\theta-1)/d\theta$$
$$=-3M\mu(r/R^3)\cos\theta\sin\theta=-3/2\;M\mu(r/R^3)\sin 2\theta$$

as found in equation (6). The negative sign results from the fact that the direction of the force is opposite to the direction in which θ is increasing, as will be apparent from a reference to figure 1.

24. It is evident from the preceding paragraph that when the potential varies from point to point over a water surface, such as the surface of the oceans, the water tends to move from areas of low potential toward areas of high potential, just as it would tend to move from areas having a higher elevation toward the areas having a lower elevation. When a water surface is in equilibrium, the total potential of all forces acting upon it evidently must be the same at all points on the surface.

25. The lunar tide-producing potential at any point P on the surface of the earth is found at once by substituting the earth's radius a for r in equation (11), and is:

$$V_t=\tfrac{1}{2}M\mu(a^2/R^3)(3\cos^2\theta-1) \qquad (12)$$

This potential is evidently a maximum at P_1 and P_2, figure 5, where $\theta=0$ and 180°, respectively, and a minimum on the great circle P_3P_4, where $\theta=90°$. The difference in the potential will therefore tend to cause the water of the oceans to pile up toward P_1 and P_2 as was shown from the analysis of the tide-producing forces in paragraph 18. To an observer at any point on the great circle

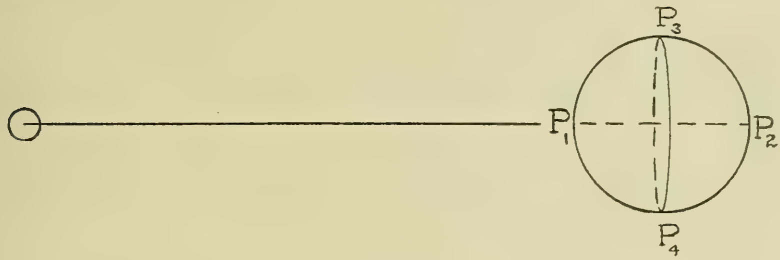

FIGURE 5.

P_3P_4 the moon is on the horizon; at P_1 directly overhead. The tide-producing potential at any point is therefore a minimum when the moon is on the horizon, and a maximum when it attains its greatest altitude above (or below) the horizon.

THE SURFACE OF EQUILIBRIUM AND THE EQUILIBRIUM TIDE

26. *Lunar equilibrium tide.*—If the earth, instead of rotating daily about its axis, rotated once in a lunar month, so that the same side of the earth was always presented to the moon, the lunar tide-producing force evidently would create two permanent bulges or distortions in the surface of the oceans, which would be directed toward the moon on one side of the earth, and in the opposite direction on the other. The surface of the oceans would then conform to the surface of equilibrium resulting from the joint action of the force of gravity and the lunar tide-producing force. If the oceans entirely covered the earth, this surface of equilibrium evidently would take the form of a prolate spheroid of revolution, with its axis pointing toward the moon, as shown in figure 6. The displacement of this theoretical tidal surface of equilibrium from the

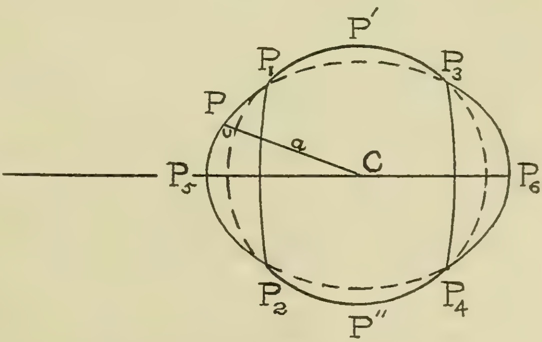

FIGURE 6.—Tidal surface of equilibrium.

spherical equilibrium surface produced by the action of gravity alone, affords a yardstick for measuring the effect of the tide-producing force of the moon, and is called the *lunar equilibrium tide.*

27. *Equation of the tidal surface of equilibrium.*—Let r be the distance CP (fig. 6) from the center of the earth to any point P on the equilibrium surface, θ the angle between CP and the axis of the surface, and as before E and M the masses of the earth and moon, respectively, R the distance between their centers, a the radius of the earth, and μ the coefficient of gravitational attraction. Let V_t and V_g be, respectively, the lunar tide-producing potential and potential due to gravity at P.

The force of gravity becomes zero when r is infinite. The gravity potential is then, from the definition in paragraph 22:

$$V_g = \int_r^\infty (E\mu/r^2)dr = E\mu/r. \tag{13}$$

Since, as shown in paragraph 24, the total potential at all points on the surface of equilibrium is constant:

$$V_t + V_g = C.$$

Substituting the expression for V_t found in equation (11), and for V_g in equation (13), the equation of the surface of equilibrium becomes:

$$\tfrac{1}{2}M\mu(r^2/R^3)\ (3\cos^2\theta - 1) + E\mu/r = C. \tag{14}$$

28. An indefinite number of surfaces are given by this equation as various values are assigned to C. If the oceans were continuous, the particular surface to be chosen would have a volume equal to that of the sphere with radius a, since the volume cannot be altered by the tidal disturbance. It will be shown that this condition is fulfilled by the surface whose radius vector is equal to the earth's radius where the tide-producing potential is zero, i. e., where $\cos^2\theta = 1/3$. Such a surface will intersect the sphere in the small circles P_1P_2, and P_3P_4 in figure 6. The resulting value of the constant is found by placing $r = a$ and $\cos^2\theta = 1/3$ in equation (14), giving:

$$E\mu/a = C$$

and the equation of the surface of equilibrium is therefore:

$$\tfrac{1}{2}M\mu(r^2/R^3)\ (3\cos^2\theta - 1) + E\mu/r = E\mu/a$$

which reduces to:

$$\tfrac{1}{2}(Ma^3/ER^3)\ (3\cos^2\theta - 1) = a^2(r-a)/r^3$$

Representing the height of the equilibrium tide by u, it follows from the definition in paragraph 26:

$$r - a = u.$$

Substituting this expression in the preceding equation,

$$\tfrac{1}{2}(Ma^3/ER^3)(3 \cos^2 \theta - 1) = a^2u/(a+u)^3. \qquad (15)$$

Since u is very small in comparison with a, the value of $(a+u)^3$ is always very close to that of a^3. Equation (15) then becomes:

$$u = \tfrac{1}{2}(Ma^3/ER^3)a(3 \cos^2 \theta - 1) \qquad (16)$$

This is the equation of the lunar equilibrium tide.

29. The volume of the tidal surface of equilibrium evidently is the same as the volume of the undisturbed sphere if the total positive tidal volume over the zones $P_1P_2P_5$ and $P_3P_4P_6$ in figure 6 is equal to the negative tidal volume over the zone $P_1P_3P_4P_2$; or, what is the same thing, if the positive and negative tidal volumes in the hemisphere $P_5P'P''$ are equal.

In figure 7, P is any point on the tidal-equilibrium surface; CP its radius vector, r; P_1P the equilibrium tide, u, at that point; CP_1 the

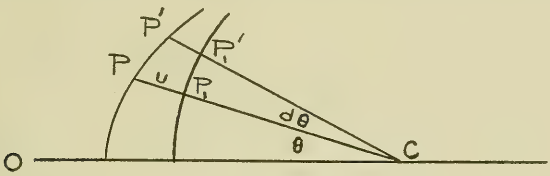

FIGURE 7.

radius, a, of the undisturbed sphere; θ its angle with the line CO directed toward the moon; and CP'_1P' the position of CP_1P when θ is increased by the differential angle $d\theta$. The equilibrium surface is, as has been seen, a surface of revolution whose axis is CO. A well known theorem establishes the volume of a solid formed by rotating a plane figure about an axis in the same plane as the product of the area of the plane figure by the length of the circumference of the circle described by its center of gravity.

The area of the differential triangle $PP'C$ is $\tfrac{1}{2} r^2\, d\theta$ and the radius of the circle described by its center of gravity is $2/3\, r \sin \theta$. The differential volume resulting from the rotation of the triangle about CO is therefore:

$$2/3\ \pi r^3 \sin \theta d\theta = 2/3\ \pi(a+u)^3 \sin \theta d\theta$$

The corresponding differential volume of the sphere is

$$2/3\ \pi a^3 \sin \theta\, d\theta$$

The difference between these volumes is the elementary tidal volume, dq, generated by the rotation of $PP_1P_1'P'$ and is:

$$\begin{aligned} dq &= 2/3\ \pi[(a+u)^3 - a^3] \sin \theta\, d\theta \\ &= 2/3\ \pi a^3(3u/a + 3u^2/a^2 + u^3/a^3) \sin \theta\, d\theta \end{aligned}$$

Since the ratio u/a is extremely small, its squares and higher powers may be dropped, giving:

$$dq = 2\pi a^2 u \sin\theta d\theta$$

Substituting the expression for u in equation (16) and integrating:

$$\begin{aligned} q &= \int \pi (Ma^6/ER^3)(3\cos^2\theta - 1)\sin\theta d\theta \\ &= \pi (Ma^6/ER^3)\left(\int 3\cos^2\theta \sin\theta d\theta - \int \sin\theta d\theta\right) \\ &= \pi (Ma^6/ER^3)(-\cos^3\theta + \cos\theta) + C \qquad (17) \end{aligned}$$

Taking $q=0$ when $\theta=0$, the constant of integration becomes zero, and the expression for the tidal volume in the zone measured by the angle θ is:

$$q = \pi (Ma^6/ER^3)(\cos\theta - \cos^3\theta) \qquad (18)$$

This volume reaches a maximum when $3\cos^2\theta - 1 = 0$ and is then

$$\pi (Ma^6/ER^3)(\sqrt{1/3} - 1/3\sqrt{1/3}) = 2/9\ \pi (Ma^6/ER^3)\sqrt{3}$$

which is the volume of the positive tide over the zone $P_5P_1P_2$ in figure 6. The volume of the negative tide is the same, as q reduces again to zero when $\theta=90°$. The condition of continuity is therefore fulfilled by the expression for the equilibrium tide given in equation (16).

30. *Magnitude of the lunar equilibrium tide.*—Assigning to the constants in equation (16) their numerical values, the ratio, M/E, of the mass of the earth to the mass of the moon is 1/81.45; a, the mean radius of the earth, 3,959 statute miles; R, the mean distance to the moon, 238,857 statute miles. The coefficient $\frac{1}{2}(Ma^3/ER^3)a$ then is 0.584 feet. The corresponding height of the lunar equilibrium tide in feet is therefore:

$$u = 0.584(3\cos^2\theta - 1)$$

The factor $3\cos^2\theta - 1$ has a maximum value of 2 when $\theta=0$, and a minimum value of -1 when $\theta=90°$. The maximum range of the lunar equilibrium tide is then $3\times0.584=1.752$ feet. This distortion of the water surface of the earth is very small in comparison with the distortion due to the earth's rotation, since the latter, as measured by the difference between the equatorial and polar radii, is 13.35 miles. The tidal distortion is however superimposed upon and not measurably affected by the distortion due to the earth's rotation.

31. *Solar equilibrium tide.*—The solar equilibrium tide is, by transposing in equation (16);

$$u_1 = \frac{1}{2}(Sa^3/ER_1^3)a(3\cos^2\theta_1 - 1)$$

The numerical value of S/E is 333,432; and the mean distance, R_1, from the earth to the sun, is 92,897,416 statute miles. The substitution of these values gives:

$$u_1 = 0.270(3 \cos^2 \theta_1 - 1)$$

The maximum range of the solar equilibrium tide is $3 \times 0.270 = 0.810$ feet.

32. *Equilibrium tide dependent on the fourth power of the moon's parallax.*—If the second term of equation (4), paragraph 15, is included in the derivation of the lunar tide-producing potential, paragraph 22, and of the lunar equilibrium tide, paragraph 28, the equation of the latter (equation 16) becomes:

$$u = \tfrac{1}{2}(Ma^3/ER^3)a(3 \cos^2 \theta - 1) + \tfrac{1}{2}(Ma^4/ER^4)a(5 \cos^3 \theta - 3 \cos \theta) \qquad (20)$$

The second term of this equation, $\tfrac{1}{2}(Ma^4/ER^4)a(5 \cos^3 \theta - 3 \cos \theta)$, is the "lunar equilibrium tide dependent on the fourth power of the moon's parallax."

Substituting the numerical values for the constants in the coefficient of this term, this part of the tide has the value, in feet, of

$$0.007(5 \cos^3 \theta - 3 \cos \theta)$$

The factor $(5 \cos^3 \theta - 3 \cos \theta)$ has a maximum value of 2 when $\theta = 0$, decreases to -0.894 when $\theta = 63°26'$, increases to 0.894 when $\theta = 116°34'$ and again decreases to a minimum of -2 when $\theta = 180°$, repeating this variation in the third and fourth quadrants. This part of the equilibrium tide is shown, on a greatly exaggerated scale, in figure 8.

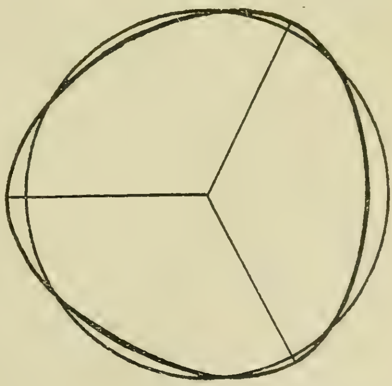

FIGURE 8.—Surface of equilibrium dependent on fourth power of moon's parallax.

It will be noted that the equilibrium tide dependent upon the fourth power of the moon's parallax goes through *three* fluctuations from maxima to minima as θ goes through a cycle from 0 to 360°; but that its maximum range is but one-quarter of an inch. It is superimposed upon and produces but an immaterial distortion of the principal equilibrium tide due to the third power of the moon's parallax, previously developed.

Since the ratio of the radius of the earth to its distance to the sun is but 1/389th of its ratio to the distance to the moon, the equilibrium tide dependent upon the fourth power of the sun's parallax is too small to be considered.

EFFECT OF ROTATION OF THE EARTH AND THE MOVEMENT OF THE MOON AND THE EARTH IN THEIR ORBITS

33. *The effect of the rotation of the earth.*—As shown in paragraph 25, the lunar and solar tide-producing forces each create two areas of high potential on the surface of the earth, one facing the moon, or sun, and the other opposite. As the earth spins around its axis, these areas make the circuit of the earth and set up the slight oscillations of the oceans which make the tides. The rise and fall of the actual tide at any locality, and the times of high water and low water, depend on the conformation of the ocean shores and beds and on the momentum of the water masses as well as on the tide-producing potential. The equilibrium tide affords a measure of the effect of tide-producing potentials alone. The variations of equilibrium tides resulting from the movements of the moon and earth in their orbits indicate the variations to be expected in the actual tides because of these movements.

34. *Effect of the declination of the moon and sun.*—The surface of equilibrium of the oceans due to the lunar tide-producing potential

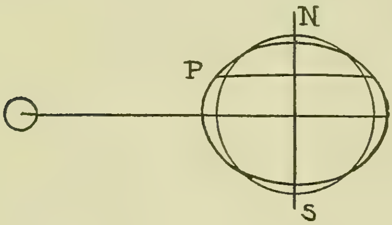

FIGURE 9.

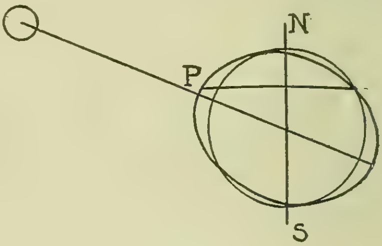

FIGURE 10.

Effect of moon's declination on tides.

has been shown to be a prolate spheroid, with its axis pointing to the moon (par. 26). When the moon is in the plane of the earth's equator, as shown in figure 9, the equilibrium tide at any point P on the earth's surface quite evidently goes through two equal fluctuations during one rotation of the earth around its axis NS, as measured from the position of the moon; i. e., two equal lunar equilibrium tides then occur each lunar day. At the earth's equator the range of these two tides is 1.75 feet when the moon is at its mean distance from the earth (par. 30), this range decreasing with the latitude of the tidal station. When, on the other hand, the moon is above or below the plane of the earth's equator (fig. 10) the two daily fluctuations of the lunar equilibrium tide quite obviously are unequal, except on the earth's equator, the inequality depending on the latitude of the tidal station, and the

angular distance of the moon from the equator. This angle is the *declination* of the moon. The solar equilibrium tides vary similarly with the declination of the sun. It may be noted that the solar equilibrium tides are equal when the days and nights have the same length, and that the inequality of the two daily tides when the moon or sun are off the equator is for a cause analogous to that of the inequality of the days and nights.

35. *Periodic variations in the declinations of the sun and moon.*—The changes in the form and range of the equilibrium tide at a given station produced by the changing declinations of the moon and sun, and the corresponding changes in the actual tides, obviously run through cycles whose respective periods are the periods of the declinations. It will be recalled that as the sun moves along the *ecliptic*, its apparent path on the celestial sphere, it crosses the celestial circle of the earth's equator, and has therefore a zero declination, at the *vernal equinox*, passing this point yearly in the latter part of March. It then ascends north of the equator and its declination reaches a maximum angle of 23°.452 at the summer solstice, late in June. This angle is the inclination of ecliptic to the equator, and may be considered as constant so far as tidal computations are concerned. At the summer solstice the sun is directly overhead at noon on the *tropic* which separates the torrid from the temperate zone in the northern hemisphere. The sun again crosses the equator at the autumnal equinox in late September, and reaches its maximum negative (south) declination of −23°.452 at the winter solstice in late December. The period of its travel from vernal equinox to vernal equinox is the *tropical year* of 365 days, 5.813 hours.

The moon, in its movement along the celestial circle marking its orbit, similarly crosses the earth's celestial equator monthly at the ascending *intersection*, reaches a maximum north (positive) declination in about a week, again crosses the equator at the descending intersection in another week, to reach its maximum south (negative) declination. The period of its travel, from ascending intersection to ascending intersection, is the *tropical month* of 27 days, 7.718 hours.

The points at which the moon's celestial orbit crosses the ecliptic are called its *ascending* and *descending nodes*, respectively. The plane of the moon's orbit has a constant inclination of 5°.145 to the plane of the ecliptic, but because of a slow retrograde movement of the moon's nodes along the ecliptic, the inclination of the moon's orbit to the equator, and hence the maximum monthly declination of the moon, slowly varies. The moon's node makes the circuit of the ecliptic in approximately 19 years.

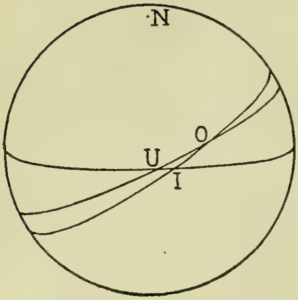

FIGURE 11.—Moon's orbit and the ecliptic.

In figure 11:

U is the vernal equinox.
O the moon's ascending node.
I the intersection of the moon's orbit with the equator.
UO the ecliptic.
UI the celestial equator.
IO the moon's orbit on the celestial sphere.

When the moon's ascending node coincides with the vernal equinox the inclination of the moon's orbit to the equator has its maximum value of $23°.452+5°.145=28°.597$. When 9½ years later it coincides with the autumnal equinox the inclination of the orbit has a minimum value of $23°.452-5°.145=18°.307$. The maximum monthly declinations of the moon, both positive and negative, range between the same limits.

36. *Longitude of the moon's node.*—The angular distance on the ecliptic from the vernal equinox U to the moon's ascending node O, figure 11, is the *longitude of the moon's node*, and is designated by the letter N. It determines the inclination, I, of the moon's orbit to the equator. The value of I may be found from N by the solution of the spherical triangle IUO, since in this triangle the angle IUO is the known inclination of the ecliptic to the equator and IOU is the known inclination of the orbit to the ecliptic. The values of I in terms of N are tabulated in manuals on tidal analysis.

37. *The lunar equilibrium tide in terms of the latitude of the tidal station and the moon's declination.*—In figure 12, CN is the axis of the earth, N its north pole, P_1M_1 the equator, the angle P_1CP the latitude, λ (lambda), of a tidal station P, NPP_1 the meridian through P, the angle M_1CM the declination, δ (delta), of the moon, $NM'M_1$ the hour circle through the line CM joining the centers of the earth and the moon, and the spherical angle PNM_1 the hour angle, H, of the moon with respect to the meridian through P. The angle PCM' is then θ, the zenith distance of the moon.

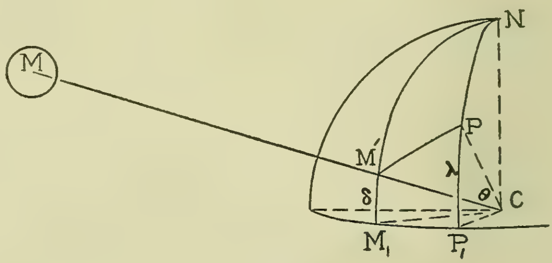

FIGURE 12.

From the spherical triangle PNM':

$$\cos\theta=\cos(90°-\lambda)\cos(90°-\delta)+\sin(90°-\lambda)\sin(90°-\delta)\cos H$$
$$=\sin\lambda\sin\delta+\cos\lambda\cos\delta\cos H \qquad (21)$$

The expression for the equilibrium tide, u, in terms of $\cos\theta$ is given in equation (16):

$$u=\tfrac{1}{2}(Ma^3/ER^3)\ (3\cos^2\theta-1)a=\tfrac{3}{2}a(Ma^3/ER^3)\ (\cos^2\theta-\tfrac{1}{3})$$

Substituting the value of $\cos\theta$ given in equation (21):

$$u=\tfrac{3}{2}a(Ma^3/ER^3)\ (\cos^2\lambda\cos^2\delta\cos^2 H+2\cos\lambda\cos\delta\sin\lambda\sin\delta\cos H+\sin^2\lambda\sin^2\delta-\tfrac{1}{3}) \qquad (22)$$

Since $\cos^2 H=\tfrac{1}{2}\ (1+\cos 2H)$, $\sin\lambda\cos\lambda=\tfrac{1}{2}\sin 2\lambda$ etc., this equation reduces to:

$$u=\tfrac{3}{4}\ a(Ma^3/ER^3)\ (\cos^2\lambda\cos^2\delta\cos 2H+\sin 2\lambda\sin 2\,\delta\cos H+\cos^2\lambda\cos^2\delta+2\sin^2\lambda\sin^2\delta-\tfrac{2}{3}) \qquad (23)$$

Substituting for $\cos^2\lambda$ and $\cos^2\delta$ in the third term their equivalents, $1-\sin^2\lambda$ and $1-\sin^2\delta$ respectively, and reducing, the equation for u becomes:

$$u=\tfrac{3}{4}a\ (Ma^3/ER^3)\cos^2\lambda\cos^2\delta\cos 2H+\tfrac{3}{4}a(Ma^3/ER^3)\sin 2\lambda\sin 2\delta\cos H+\tfrac{1}{4}\ a(Ma^3/ER^3)\ (1-3\sin^2\lambda)\ (1-3\sin^2\delta) \qquad (24)$$

38. *Semidiurnal and diurnal parts of the lunar tide.*—Equation (24) shows that the lunar equilibrium tide at any tidal station is composed of the following parts:

(a) That represented by the term $\tfrac{3}{4}\ a(Ma^3/ER^3)\cos^2\lambda\cos^2\delta\cos 2H$. Since the angle $2H$ obviously goes through two complete cycles from 0 to 360° while H is making one cycle in a lunar day, this part goes through two cycles every lunar day and is therefore called the *semidiurnal part* of the tide.

(b) That represented by the term $\tfrac{3}{4}\ a(Ma^3/ER^3)\sin 2\ \lambda\sin\delta\cos H$. This part goes through one cycle each lunar day and is the *diurnal* part of the tide.

(c) That represented by the term:

$$\tfrac{1}{4}\ a(Ma^3/ER^3)(1-3\sin^2\lambda)\ (1-3\sin^2\delta).$$

Since this term is independent of the angle H, it undergoes no change because of the rotation of the earth. It is therefore the height of the daily mean sea level above that of a sea undisturbed by tidal forces. Its variation due to the changing declination of the moon will be later discussed (par. 42).

39. A typical example of the diurnal and semidiurnal fluctuations of the lunar equilibrium tide, and of the total tidal fluctuation resulting therefrom (disregarding the variation due to the changing distance between the moon and the earth) is illustrated in figure 13, which shows these fluctuations at a station at 40° north latitude during a lunar day in which the declination of the moon increases from 7° to 13°.

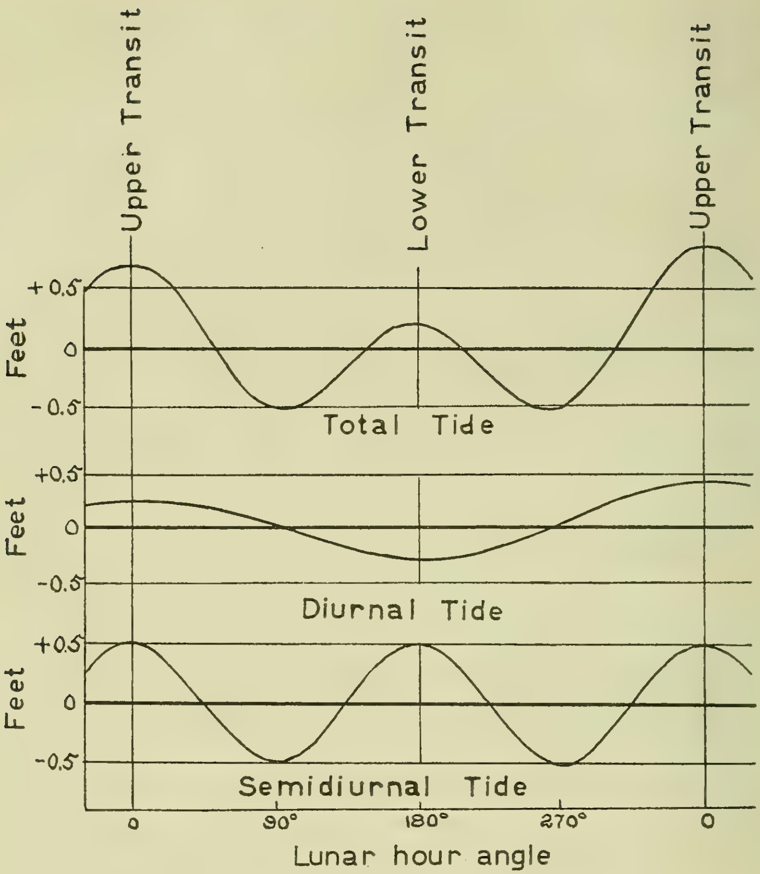

FIGURE 13.—Diurnal, semidiurnal, and resultant tides.

It will be noted that the semidiurnal part of the lunar equilibrium tide follows the sinusoidal curve of the cosine function, with a very slight skew because of the change in the moon's declination during the day. The skew of the diurnal part of the tide because of this change is more pronounced. The combination of these two parts produces the inequalities of the daily tides inferred in paragraph 34. The ranges of the two parts of the equilibrium tide are affected differently by the latitude of the tidal station. At high latitudes the diurnal part may mask the semidiurnal fluctuations and produce an equilibrium tide which is predominantly or wholly diurnal.

40. *Variations in the amplitude of the diurnal and semidiurnal parts of the lunar equilibrium tides with the moon's declination.*—The amplitude of the fluctuations of the semidiurnal part of the lunar equilibrium tide evidently varies with the coefficient of cos $2H$ in the first term of equation (24). At a tidal station whose latitude is λ, it consequently varies with $\cos^2 \delta$. It is therefore a maximum when the

declination of the moon is zero, as the moon crosses the ascending intersection of the moon's orbit with the celestial Equator. The amplitude then decreases to a minimum when, a week later, the moon has its maximum north declination; again increases to a maximum when, in another week, the moon is at its descending intersection; and decreases to a second minimum when the moon reaches its maximum south declination. The amplitude of the fluctuations of the semidiurnal part of the lunar equilibrium tide therefore varies between a maximum and a minimum twice during a tropical month.

The amplitude of the fluctuations of the diurnal part of the lunar equilibrium tide at any tidal station (except those on the earth's Equator) varies with sin 2 δ. It therefore increases twice during the tropical month from zero, when the moon has a zero declination, to a maximum when the moon has its maximum declination north or south of the Equator.

The amount of the variation in the amplitudes of the fluctuations both of the semidiurnal and the diurnal parts of the lunar equilibrium tide slowly changes with the inclination of the moon's orbit to the Equator (par. 35) and hence with the longitude of the moon's node (par. 36).

41. *Variations in the range of the actual tide with the moon's declination.*—Since the equilibrium tides are a measure of the astronomical causes of the actual tides, it may be expected that the part of the actual tide due to the moon is made up of diurnal and semidiurnal elements, each varying with the declination of the moon in the same manner manner as the equilibrium tides; the amount of the variation slowly changing with the longitude of the moon's node. It does not follow however that the diurnal and semidiurnal parts of the actual tides change with the latitude in the same manner as the parts of the equilibrium tide; for the latter, while affording a measure of the astronomical causes of the variation in the tide at a particular station, afford no indication of the relationship between the tides at two different stations.

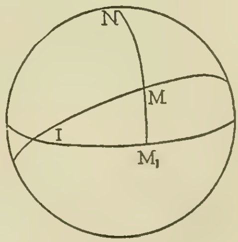

FIGURE 14.

42. *Lunar fortnightly tide.*—In figure 14, IM_1 is the celestial Equator, IM the celestial circle of the moon's orbit, I the intersection, M the position of the moon at any time, NMM_1 the hour circle through M. M_1M is then the moon's declination, δ. Let IM, the angular distance of the moon from the intersection, be represented by l. Then in the right spherical triangle IM_1M:

$$\sin \delta = \sin l \sin I \qquad (25)$$

where I is the inclination of the moon's orbit to the Equator.

Substituting this expression for sin δ in the last term of equation (24), $\frac{1}{4}\, a\, (Ma^3/ER^3)\ (1-3 \sin^2 \lambda)\ (1-3 \sin^2 \delta)$, this becomes

$\frac{1}{4} a\,(Ma^3/ER^3)\,(1-3\sin^2\lambda)\,(1-3\sin^2 l\sin^2 I)$. Again substituting for $\sin^2 l$ its equivalent, $\frac{1}{2}\,(1-\cos 2l)$ the term reduces to the two terms:

$$\begin{aligned}&\tfrac{3}{8}\, a\,(Ma^3/ER^3)\,(1-3\sin^2\lambda)\sin^2 I\cos 2l\\&+\tfrac{1}{4}\, a\,(Ma^3/ER^3)\,(1-3\sin^2\lambda)\,(1-3/2\sin^2 I)\end{aligned}$$

The first of these terms evidently goes through two cycles from 0 to 360° while l is making one cycle in the tropical month. It is called the *lunar fortnightly component* of the tide.

The last term remains constant as the earth revolves about its axis and the moon changes its declination. It represents therefore a permanent distortion of the ocean surface, so far as these movements are concerned.

The substitution of the numerical of the constants in the expression for the lunar fortnightly equilibrium tide shows that, at the earth's Equator, its range varies from 0.086 foot when the inclination of the moon's orbit is a minimum, to 0.20 foot when the inclination is a maximum. The range decreases to zero at a latitude of 35°16′ north or south of the Equator, increasing again toward the poles. The actual fortnightly tide is correspondingly small. At most tidal stations this component produces, however, a fortnightly fluctuation of an inch or more in the daily mean height of the sea.

43. *Effect of eccentricity of the moon's orbit.*—As the moon travels its orbit, its distance, R, from the earth varies. The point at which it is nearest the earth is its *perigee*, and the point at which it is the most distant is its *apogee*. Because of the disturbing effect of the attraction of the sun, the moon's orbit varies somewhat, but its distance from the earth at apogee ordinarily exceeds by more than 10 percent the distance at perigee. The moon makes the circuit from perigee to perigee in the *anomalistic* month of 27 days, 13.309 hours. Since the coefficients of the terms in equation (24) each contain the factor $1/R^3$, the amplitudes of the diurnal and semidiurnal parts of the lunar equilibrium tides, as derived from these terms, tend to vary from a maximum to a minimum once during the anomalistic month, as well as varying twice during the tropical month because of the changing declination of the moon. A corresponding variation may be expected in the actual tides.

To gage the amount of the variation in the tides because of the elliptical form of the moon's orbit, let P be the distance of the moon at perigee and $A=P+d$ its distance at apogee. The ratio of the amplitude of the equilibrium tides at perigee to those at apogee is then

$$\begin{aligned}(\tfrac{1}{2}Ma^4/EP^3)/(\tfrac{1}{2}Ma^4/EA^3)&=A^3/P^3=(P+d)^3/P^3\\&=1+3\,d/P+3\,(d/P)^2+(d/P)^3\end{aligned}\tag{26}$$

Neglecting the squares and cubes of the ratio d/P, it is apparent that the variation in the equilibrium tide is three times the variation in the distance to the moon. As the ratio d/P usually exceeds 10 percent, the amplitudes of the diurnal and semidiurnal lunar equilibrium tides are subject to a variation of 30 percent or more during the anomalistic month. This variation is sometimes called the *parallax inequality*, since the distance to the moon is measured by its *parallax* (par. 15).

The lunar fortnightly component of the tide is so small that its parallax inequality is not taken into consideration. The variation in R produces, however, a variation in the fixed term:

$$\tfrac{1}{4}\, a\, (Ma^3/ER^3)/(1-3\, sin^2\, \lambda)\, (1-\tfrac{3}{2}\sin^2 I)$$

developed in paragraph 42, giving rise to a small *monthly* tidal component with the period of an anomalistic month.

44. *The solar equilibrium tides.*—The equilibrium tide due to the sun is similarly made up of a semidurnal part, which goes through two complete cycles in a mean solar day of 24 hours; a diurnal part which goes through one cyle per day; both of which vary with the declination of the sun; together with a *semiannual* component of relatively small range. Since the eccentricity of the earth's orbit around the sun is much less than the eccentricity of the moon's orbit, the *parallax inequalities* of the solar equilibrium tides are small in comparison with those of the lunar equilibrium tides.

45. As will be shown in the following chapter, the varying semidiurnal and diurnal fluctuations of the tide, because of the changing declinations of the sun and the moon, and the varying distances of the earth from these bodies, may be resolved into *components* of fixed amplitudes with periods not far from 12 hours and 24 hours respectively.

THE ACTUAL TIDES

46. The actual fluctuations of the surfaces of the oceans because of the tide-producing forces are somewhat akin to the slopping around of the water in a basin on a moving train. The momentum of the moving masses in the deep seas tends to pile up the water in the shallow depths along the coasts, producing tides whose ranges may greatly exceed the range of the equlibrium tide. The tidal ranges at different points along the shores, and the times of high and low water, depend upon the contour of the ocean beds and the conformation of the coasts and cannot be determined by abstract calculation. The relation of the times of the actual to the equilibrium tides may possibly be more apparent if the varying pull upon the waters of the oceans due to the attraction of the moon and sun be conceived to be replaced

by the varying push that would come from a heaving of the beds of the oceans of the magnitude and sequence of the equilibrium tides. Since the equilibrium tides move across the oceans at the same speed as the moon and the sun, or upward of 660 miles per hour in the range of latitudes of the United States, the actual tides generally lag behind the equilibrium tides by a substantial interval of time. But while the range and the times of high and low water vary widely from those of the equilibrium tide, the *periods* of the fluctuation of the components of the actual tide must clearly be exactly the same as the periods of the forces which cause them, and consequently conform exactly to the periods of the equilibrium tides. Since however the oceans may be expected to respond differently to the diurnal fluctuations of the tide producing forces than to the semidiurnal fluctuations, the diurnal and semidurnal elements of the actual tides are generally displaced with respect to the respective timing of these elements of the equilibrium tides. Similar displacements may be expected in the various components into which the diurnal and semidurnal tides may be resolved. Last of all the ranges of the various components of the actual tides bear a definite relationship to the range of the like components equilibrium tides, in that small components of the latter will produce small fluctuations in the actual tide; and the variations of the various components at any tidal station, due to the longitude of the moon's node (par. 36) correspond to the variations in the equilibrium tide.

47. *Meteorological tides.*—Besides the systematic fluctuations due to the tide-producing forces, irregular fluctuations of the oceans are caused by winds and the varying barometric pressure over their surface. These accidental fluctuations are called *meteorological* tides.

48. *Examples of tides.*—The recorded tides at a few representative places are shown in figure 15. It is apparent that these tides vary widely as to type. The effect of the diurnal tidal variations in the tide at San Francisco and Galveston may be especially noted.

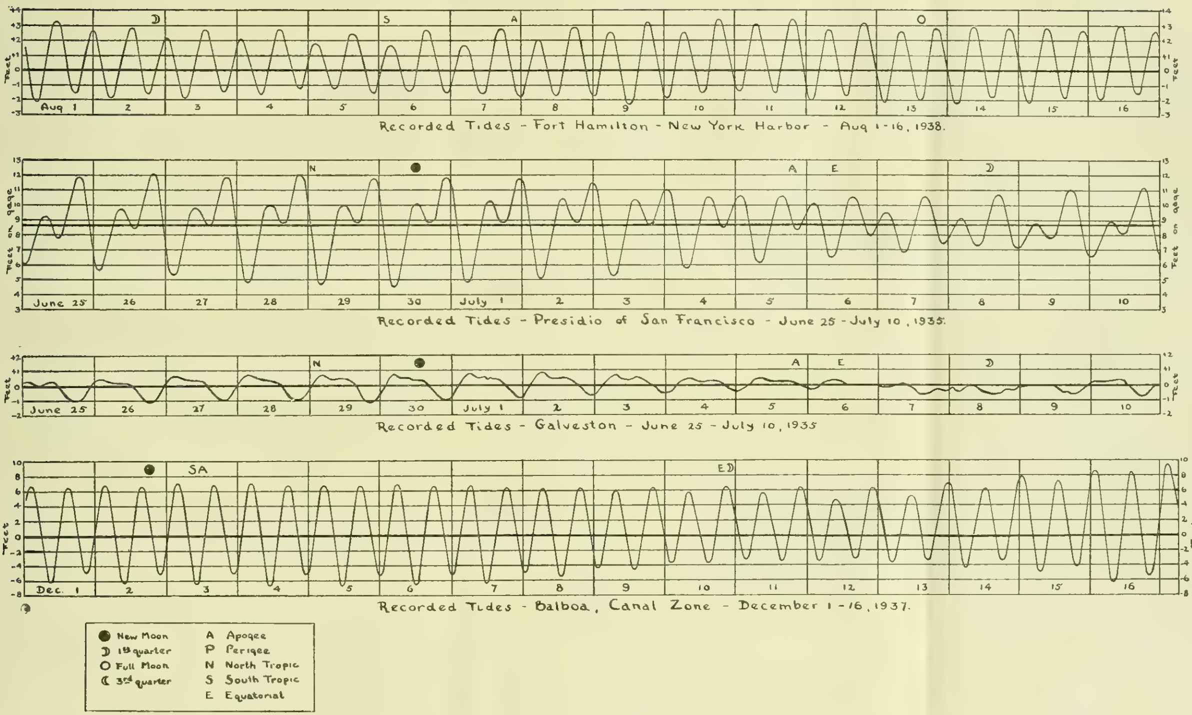

FIGURE 15.—Typical tide curves.

Chapter II

HARMONIC ANALYSIS OF THE TIDES

49. *Harmonic components of the tide.*—It is reasonable to assume, and will later be shown, that, except as affected by irregular meteorological disturbances, the tide curve at any station is the resultant of a limited number of sinusoidal (cosine or sine) curves, whose periods are determined by the periods of the tide-producing forces. This relation is expressed by the equation:

$$y=H_0+A_1 \cos (a_1t+\alpha_1)+A_2 \cos (a_2t+\alpha_2)+A_3 \cos (a_3t+\alpha_3)+ \cdot \cdot \cdot \quad (27)$$

in which y is the height, at the time t, of the tide above an arbitrarily chosen datum, H_0 is the height of mean sea level above this datum, and the subsequent terms in the form $A \cos (at+\alpha)$ are the component tides. Of each component the coefficient A is the *amplitude* or semirange, a is the *speed*, the angle $at+\alpha$ is the *phase* at the time t, and α (alpha) is the *initial phase*. Placing:

$$a=360°/T \quad \text{or } aT=360° \quad (28)$$

it follows that at increases from zero to 360° as t increases from zero to T; and again as t increases from T to $2T$, and so on. T is therefore the *period* in which the component goes through its cycle of fluctua-

tion. From the discussion in paragraph 46 it is clear that the speed, a, of each component is determined by the astronomical movements of the moon or sun, or both. The amplitude A, and the initial phase α may be determined from the recorded tides at the place, by a method hereinafter described. In the ensuing discussion, the amplitude A will be expressed in feet; the time t and the period T, in mean solar hours; the speed, a, in degrees per hour (unless otherwise indicated) and the initial phase α in degrees.

Each component is graphically represented by the projection on the Y axis (fig. 16), of a generating radius CP, of length equal to the

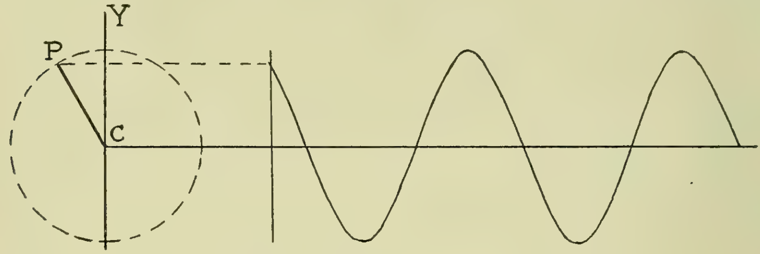

FIGURE 16.—Generating radius and tide curve of a component.

amplitude of the component, rotating at the constant *speed* of the component around the origin, C, its initial angle with the Y axis being the *initial phase* of the component.

The direction of the rotation of the generating radius is taken as positive in a counterclockwise direction, in accordance with the usual trigonometric convention.

The graph of the component is the sinusoidal cosine curve shown on the right in figure 16, in which the abscissas represent time and the ordinates the height of the component above mean sea level.

It is sometimes more convenient to write equation (27) in the form:

$$y=H_0+A_1\cos(a_1t-\zeta_1)+A_2\cos(a_2t-\zeta_2)+A_3\cos(a_3t-\zeta_3)+\cdots \quad (29)$$

in which the angles designated as ζ (zeta) are numerically equal to the respective initial phases, α, of the several components but opposite in sign. Since each component reaches its maximum when $at-\zeta=0$, and hence when $t=\zeta/a$, it is evident that ζ/a is the time of high water of the component next after the origin of time.

50. *"Astres Fictifs."*—The tide represented by a single component would have high waters and low waters of constant heights occurring at equal intervals of time. Such a tide would be generated by a moon traveling at a constant angular speed along a circular orbit in the plane of the earth's Equator. Each component of the tide is therefore sometimes treated as the tide due to a fictitious moon, or "astre fictif," moving at a uniform speed along the earth's celestial

Equator. This conception appears, however, to complicate rather than to simplify a consideration of the tidal components and is not herein pursued.

51. *Combination of components.*—A consideration of the manner in which two or more components combine with each other will afford a basis for the selection of the speeds of the particular components which reproduce the tide at any station as it varies with the changing positions and distances of the moon and sun.

Taking the two components:

$$y_1 = A_1 \cos (a_1 t + \alpha_1) \qquad y_2 = A_2 \cos (a_2 t + \alpha_2) \tag{30}$$

let CP_1 and CP_2 (fig. 17) be the positions of their generating radii at

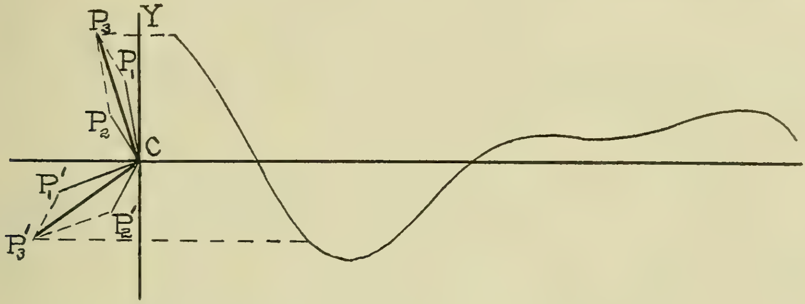

FIGURE 17.—Resultant of components of unequal speeds.

any instant of time. Completing the parallelogram $CP_1P_3P_2$ it is at once apparent that the projection on the Y axis of the resultant vector CP_3 is the algebraic sum of the ordinates of the two components y_1 and y_2 at that instant. This resultant vector CP_3 will therefore generate the tide curve of the resultant of the two components, as shown on the right of the figure.

If CP_1, CP_2, and CP_3 (fig. 18), are the generating radii of three components at any instant, the length and position of the resultant vector is given by the line CP, found by drawing P_1P_2 parallel and equal to CP_2, and $P_2\ P$ parallel and equal to CP_3. The resultant vector of any number of components may be drawn in a similar manner.

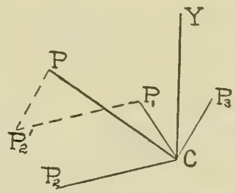

FIGURE 18.—Resultant vector of three components.

52. *Two components of the same speed.*—If two components have the same speed, the angle between the generating radii CP_1 and CP_2 (fig. 17), remains constant, the parallelogram $CP_1P_3P_2$ does not change its shape as the radii rotate around C, and the resultant vector CP_3 therefore remains of constant length and rotates at the constant speed of the two components. It will therefore generate a sinusoidal curve as shown in figure 19.

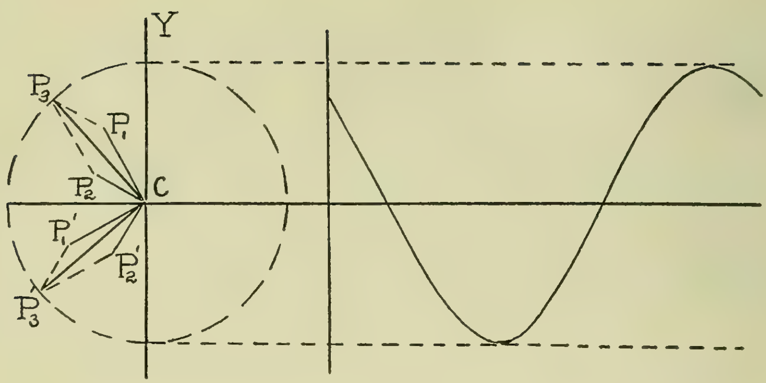

FIGURE 19.—Resultant of two components of the same speed.

Designating the common speed of the two components by a, the constant length of the resultant radius by A_3, and its initial angle with CY (when $t=0$) by α_3, it follows that

$$A_1 \cos (at+\alpha_1)+A_2 \cos (at+\alpha_2)=A_3 \cos (at+\alpha_3) \qquad (31)$$

It will be seen therefore that any two components having the same speed unite into a component of the identical speed.

53. *Two components of different speeds.*—If two components have different speeds, the faster of the two generating radii CP_1 or CP_2 is continuously gaining on the slower, the angle between these radii progressively changes, the parallelogram $CP_1P_3P_2$ steadily changes its form, and the length of the resultant vector CP_3, together with its speed, varies with the time. The curve generated by the resultant vector takes various forms, depending on the amplitudes and speeds of the components; but the periodic variations in the length and speed of the resultant vector only need be here considered.

54. *Variation in the length of the resultant vector.*—When the amplitudes of the two components differ, A_1 may be taken as the amplitude of the major component, a_1 its speed; and A_2 the amplitude of the minor component. Let b be the algebraic difference between the speeds of the components, so that $a_2=a_1+b$. The faster of the generating radii of the two components evidently will overtake and pass periodically the slower. Taking for convenience the origin of time at a moment when the generating radii coincide, the two components then have the form

$$y_1=A_1 \cos a_1t \qquad y_2=A_2 \cos (a_1+b)t \qquad (32)$$

In figure 20, CP_1 and CP_2 are the positions of the generating radii of the two components at any time t, and CP_3 the resultant. Since the angle YCP_1 represents a_1t and the angle YCP_2 represents $(a_1+b)t$, the angle between the radii, P_1CP_2, is $(a_1+b)t-a_1t=bt$.

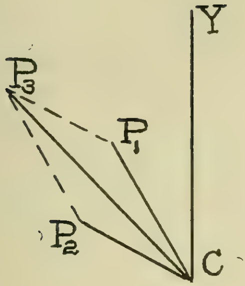

FIGURE 20.

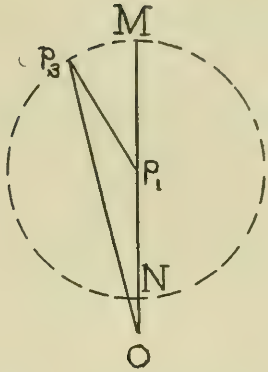

FIGURE 20 A.

Variation in length of resultant vector.

If then in figure 20A, Op_1 is laid off equal to A_1, the angle Mp_1p_3 equal to bt, and p_1p_3 equal to A_2, the triangle Op_1p_3 in figure 20A reproduces the triangle CP_1P_3 in figure 20, and Op_3 is the length of the resultant vector CP_3. As the time increases the point p_3 evidently describes a circle of radius A_2 around p_1 as a center. The length of the resultant vector fluctuates between $OM=A_1+A_2$ and $ON=A_1-A_2$. The period in which the point p_3 completes the circuit around p_1 as a center obviously is $360°/b$. This interval is called the *synodic period* of the two components, and, without regard to the algebraic sign of b, is the interval between the successive times at which the resultant vector reaches its maximum length, and also between the times at which this vector has its minimum length.

55. *Variation in the speed of the resultant vector.*—A consideration of figure 20A makes it apparent that the resultant vector alternately leads and lags behind the radius vector of the major component; and that its mean speed is the speed of the major component. A study of the figure shows further that when b is positive, the point p_3 rotates around p_1 in the same direction that CP_1 rotates around C, and the speed of the resultant is a maximum at the point M, when the length of the resultant is a maximum. When b is negative, p_3 rotates in the opposite direction and the speed of the resultant is a maximum at the point N, when the length of the resultant is a minimum.

56. *Speed of the resultant of two components of equal amplitudes.*—If the amplitudes of the two components are equal, the resultant CP_3 (fig. 20), evidently bisects the angle P_1CP_2, and the angle YCP_3 is therefore equal to $(a_1+\frac{1}{2}b)t$. The speed of the resultant therefore has the constant value of $a_1+\frac{1}{2}b$, the average of the speeds of the two components.

57. *Form of resultant of two components whose speeds are nearly equal.*—If the difference between the speeds of two components is relatively small, so that the length of their resultant vector changes

little during one revolution of the component radii, the curve representing the resultant of the two components evidently takes the general form of a sinusoidal curve, with an amplitude slowly fluctuating between the sum and difference of the amplitudes of the components, as shown in figure 21.

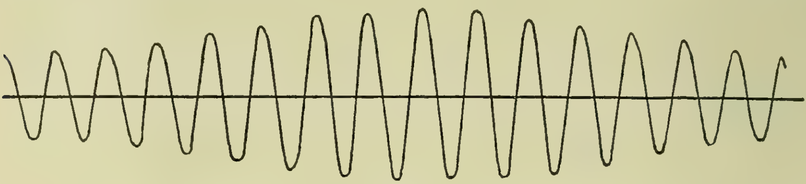

FIGURE 21.—Tide curve of two components whose speeds are nearly equal.

If A_1 is the amplitude of the major component and b the numerical difference between the speeds of the two components, the equation of the resultant has the form:

$$y = A_1 \cos (a_1 t + \alpha_1) + A_2 \cos [(a_1 + b)t + \alpha_2] \tag{33}$$

when the speed of the minor component is the greater, and:

$$y = A_1 \cos (a_1 t + \alpha_1) + A_2 \cos [(a_1 - b)t + \alpha_2] \tag{34}$$

when the speed of the minor component is less than that of the major. From the discussion in paragraph 55 it is apparent that in the first case, (equation 33), the speed of the resultant is greater than that of the major component when the amplitude of the resultant is large, and less when it is small. The high and low waters of the resultant shown in figure 21 will then progressively lead those of the major component when the amplitude of the resultant is large, and progressively drop back again when its amplitude is small. In the second case (equation 34), the high and low waters of the resultant will progressively lag behind those of the major component when the amplitude of the resultant is large, and progressively catch up with them when the amplitude is small. By taking the sum of the three components:

$$y = A_1 \cos (a_1 t + \alpha_1) + A_2 \cos [(a_1 + b)t + \alpha_2] + A_3 \cos [(a_1 - b)t + \alpha_3] \tag{35}$$

the timing of the high and low waters of the resultant may be made, by the selection of the relative values of A_2 and A_3, to conform to a systematic variation from the timing of the high and low waters of the principal component. If A_2 is equal to A_3, the timing of the high and low waters of the resultant will conform exactly to those of the simple harmonic component $A_1 \cos (a_1 t + \alpha)$, since the speed of the resultant of the last two terms in equation (35) is, as shown in paragraph 56:

$$\tfrac{1}{2} [(a_1 + b) + (a_1 - b)] = a_1. \tag{36}$$

58. If the amplitudes of two components are equal, $A_1-A_2=0$ and if their speeds are nearly equal, the curve of the resultant takes the form shown in figure 22.

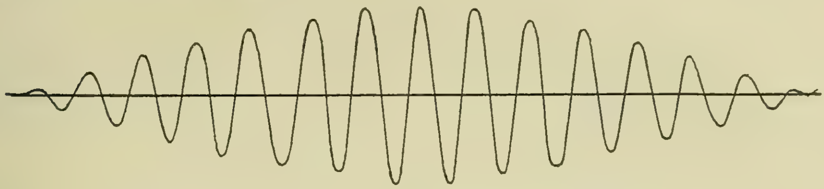

FIGURE 22.—Tide curve of two components of equal amplitudes.

Since the speed of the resultant in this case is the average of the speeds of the two components, the equation of the resultant conveniently may be written:

$$y=A \cos [(a+\tfrac{1}{2}b)t+\alpha_1]+A \cos [(a-\tfrac{1}{2}b)t+\alpha_2] \qquad (37)$$

where a is the speed of the resultant and b the difference in the speeds of the two components.

59. The form of the curves shown in figures 21 and 22 indicates the manner by which the periodical variations in the semidiurnal and diurnal parts of the tidal fluctuations at any station, due to the changing declinations and distances of the moon and sun (pars. 40–44), may be represented by a combination of components of properly chosen speeds, fixed by the accurately established periods and speeds of the movements of the moon and sun.

ASTRONOMICAL PERIODS DETERMINING THE SPEEDS OF TIDAL COMPONENTS

60. *Mean solar and lunar days.*—The intervals between the successive transits of the true sun across the meridian of a place vary slightly with its declination and distance, increasing as its declination increases either north or south of the equator, and increasing also as its distance decreases. The mean interval is the *mean solar day* of 24 mean solar hours. The speed at which the hour angle of the true sun increases obviously must be the least when the interval between its transits is the greatest. The speed of the hour angle of the true sun, in degrees per mean solar hour, as influenced by the declination, is therefore a maximum when the declination is zero, and a minimum when the declination is at its maximum. As influenced by the distance, the speed is a minimum at *perihelion*, when the distance to the sun is the least, and a maximum at *aphelion*, when the distance is the greatest. The angular speed of the hour angle of the moon goes through similar but more rapid changes, with further disturbances because of the attraction exerted by the sun on the moon. The mean

interval between the upper, or visible, transits of the moon across the meridian of a place is the *mean lunar day* of 24.841,202,4 mean solar hours.

61. It may be noted that the slow tilting to and fro of the plane of the moon's orbit with respect to the plane of the earth's Equator because of the changing longitude of the moon's node (par. 36) does not affect the length of the *mean* lunar day, the *mean* speed of the lunar hour angle, or the mean period of the travel of the moon from intersection to intersection. The intersection moves to and fro along the Equator in a small arc on either side of the equinox, its mean position being the equinox itself.

62. The mean periods and the speeds, or mean angular changes per mean solar hour, of the movements of the sun and moon pertinent to the development of the speeds of the tidal components are as follows:

TABLE I

Cycle	Period in mean solar hours	Speed in degrees per solar hour
Mean solar day	24	15
Mean lunar day	24.841,202,4	14. 492, 052, 1
Tropical month—moon's travel from intersection to corresponding intersection	655. 717, 96	. 549, 016, 5
Anomalistic month—moon's travel from perigee to perigee (par. 43)	661. 309, 20	. 544, 374, 7
Synodic month—full moon to full moon	708. 734, 1	. 507, 947, 9
Tropical year—sun's travel, equinox to corresponding equinox	8765. 812, 7	. 041, 068, 6
Anomalistic year—sun's travel, perihelion to perihelion	8766. 230, 9	. 041, 066, 7

63. The "speeds" in the preceding tabulation are the quotients of 360° divided by the corresponding period, in accordance with the definition expressed in equation (28). The speed of the mean lunar day is the mean hourly change in the angle between the hour circle through the moon and the meridian of the place. The speed of the tropical month is the mean hourly change in the angle between the hour circle through the moon and that through the intersection of the moon's orbit with the Equator. The sum of these two speeds is therefore the mean hourly change between the meridian and the intersection. Similarly the sum of the speeds of the solar day and the tropical year is the mean hourly change between the meridian of the place and the equinox. Since the mean position of the intersection is at the equinox, the two sums are the same, i. e.,

$$\begin{array}{rr} 15.000,000,0 & 14.492,052,1 \\ .041,068,6 & .549,016,5 \\ \hline 15.041,068,6 & 15.041,068,6 \end{array}$$

SPEEDS OF THE TIDAL COMPONENTS

64. *Semidiurnal lunar components.*—It has been shown in paragraphs 40 and 41 that the amplitude of the semidiurnal part of the lunar tide increases and decreases twice during the tropical month because of the changing declination of the moon, and in paragraph 43 that it increases and decreases once during the anomalistic month because of its changing distance. Since the effect of one of these variations on the other is small, these variations may be reproduced by a combination of a major component and two pairs of minor components in the form indicated in paragraph 57; i. e.,

$$y=\mathrm{M}_2 \cos\ (\mathrm{m}_2 t+\alpha_1)+\mathrm{K}_2 \cos\ [(\mathrm{m}_2+b)t+\alpha_2]+\mathrm{K}_2' \cos[(\mathrm{m}_2-b)t+\alpha_3]$$
$$+\mathrm{L}_2 \cos\ [(\mathrm{m}_2+c)t+\alpha_4]+\mathrm{N}_2 \cos\ [(\mathrm{m}_2-c)t+\alpha_5] \qquad (38)$$

In equation (38), M_2 is the amplitude of the major component. Its speed m_2 is the mean speed of the semidiurnal component of the tide, and is therefore twice the speed of a lunar day, or 28.°984,104,2 per mean solar hour (par. 55).

Either or both of the next two terms will produce a variation in the amplitude of the resultant, of the same period as the variation in the amplitude of the actual tide due to the changing declination of the moon, if $360°/b$ is made equal to that period (par. 54). Since the period of these fluctuations is one half of a tropical month, b is twice the "speed" of the tropical month, or 1.°098,033,0 per mean solar hour. The relative amplitudes of this pair of minor components should produce a variation in the speed of the resultant conforming to the variation in the speed of the semidiurnal tide, and hence in the true speed of the hour angle of the moon, due to the changing declination of the moon. Since, when the effect of the varying declination is alone considered, the hour angle reaches its maximum speed when the declination is zero (par. 60) and the amplitude of the semidiurnal part of the lunar equilibrium tide is then a maximum (par. 40), the component with the greater speed, $\mathrm{K}_2 \cos\ [(\mathrm{m}_2+b)t+\alpha_2]$ must be the dominant one of the pair (par. 57). A mathematical derivation of the tidal components, later outlined, shows that this component correctly reproduces the entire variation in the amplitude of the semidiurnal part of the lunar equilibrium tide, and hence of the actual tide as well, because of the changing declination of the moon. The third term in equation (38) therefore disappears.

Similarly either or both of the fourth and fifth terms of equation (38) will produce a variation in the amplitude of the resultant of the same period as that of the actual tide due to the changing distance of the moon if $360°/c$ is made equal to the anomalistic month, or if c is the speed of the anomalistic month, 0.°544,374,7 per mean solar hour.

Since the speed of the hour angle of the moon tends to become a minimum at perigee, when the increase in the amplitude of the semidiurnal part of the lunar equilibrium tide because of the decreased distance of the moon is a maximum, the component with the lesser speed, $N_2 \cos [(m_2-c)t+\alpha_5]$, is the larger. Both of this pair of components are found necessary, however, to represent the variations in the tide because of the changing distance of the moon.

65. Writing then equation (38) in the form:

$$y=M_2 \cos (m_2t+\alpha_1)+K_2 \cos (k_2t+\alpha_2)+L_2 \cos (l_2t+\alpha_3)N_2 \cos (n_2t+\alpha_4) \tag{39}$$

$m_2=28°.984,104,2$ per mean solar hour.
$k_2=m_2+b=30°.082,137,2$ per mean solar hour.
$l_2=m_2+c=29°.528,478,7$ per mean solar hour.
$n_2=m_2-c=28°.439,729,5$ per mean solar hour.

The detailed mathematical analysis shows that components having these speeds are sufficient to reproduce with substantial accuracy the lunar semidiurnal part of the tide. Certain other small components are developed by that analysis, principally to account for the variations in the tides due to the irregularities in the movement of the moon because of the sun's attraction, but these are of no substantial importance.

66. *Designation of components.*—The capital letters designating the amplitudes of the components whose speeds are identified in the preceding paragraphs are those conventionally assigned to these components. The subscript 2 indicates that the component is a semidiurnal one; i. e., that its speed is in the vicinity of 30° per mean solar hour. Diurnal components, with a speed not far from 15° per hour, are given the subscript 1. The speed of the component is conventionally designated by the corresponding small letter of the alphabet, and is given the same subscript. Components are customarily referred to by the letter and subscript designating the amplitude; i. e., as the "M_2 component," and "K_2 component," etc. The components previously identified are named as follows:

M_2, *the principal lunar component,*
N_2, *the larger lunar elliptic, semidiurnal,*
L_2, *the smaller lunar elliptic, semidiurnal,*
K_2, the discussion in paragraph 63 indicates that a solar component of the same speed is to be anticipated. This component is therefore called the *lunar portion of the lunisolar semidiurnal.*

67. *Solar semidiurnal components.*—The speeds of the solar semidiurnal components corresponding to the lunar semidiurnal components already identified may be written at once by substituting the

solar for the lunar day and the tropical year for the tropical month. These components are:

S_2, *the principal solar.* Since its speed is twice that of a mean solar day, $s_2=30°$ per hour.

T_2, *the larger solar elliptic, semidiurnal.* Its speed is the difference between that of the principal solar and that of the anomalistic year and is therefore $t_2=30-0°.041,066,7=29°.958,933,3$ per hour.

R_2, *the smaller solar elliptic, semidiurnal.* Its speed is the sum of that of the principal solar and of the anomalistic year, and is therefore $r_2=30+0°.041,066,7=30°.041,066,7$ per hour.

K_2, *the solar portion of the lunisolar semidiurnal.* Its speed is twice the sum of the speeds of the solar day and the tropical year, and is therefore $30°.082,137,2$ per hour.

Since the eccentricity of the earth's orbit around the sun is relatively small, the T_2 and R_2 components are small in comparison with the N_2 and L_2 components respectively. Since the lunar and solar parts of the lunisolar semidiurnal components have the same speed, they unite into a single component of that speed (par. 52), designated as the *lunisolar semidiurnal component* K_2.

68. *Lunar diurnal components.*—The amplitude of the lunar diurnal part of the tide has been shown to increase from zero to a maximum and back again to zero twice during a tropical month because of the changing declination of the moon. This part of the tide follows therefore a curve of the characteristic form shown in figure 22. Such a curve is represented by the sum of the two components:

$$y=A\cos\left[(a+\tfrac{1}{2}b)t+\alpha_1\right]+A\cos\left[(a-\tfrac{1}{2}b)t+\alpha_2\right] \tag{40}$$

in which a is the speed of the resultant of the two components, and $360°/b$ is the period of the fluctuation of the resultant (par. 58). The speed of the resultant lunar diurnal tide is the speed of the lunar day, designated as m_1. If T is the period of the tropical month

$$360°/b=\tfrac{1}{2}T$$

whence

$$360°/\tfrac{1}{2}b=T$$

It follows from the definition in paragraph 63 that $\frac{1}{2}b$ is the "speed" of the tropical month.

The data given in paragraph 62 show that the numerical values of the speeds of the two components in equation (40) are respectively:

$$m_1+\tfrac{1}{2}b=14°.492,052,1+0°.549,016,5=15°.041,068,6$$

$$m_1-\tfrac{1}{2}b=14°.492,052,1-0°.549,016,5=13°.943,035,6$$

The first of these speeds is one half of the speed of the lunisolar semi-diurnal component K_2. It is therefore designated as k_1. The second is conventionally designated as o_1.

Since the amplitude of each of the lunar diurnal components just developed also fluctuates during the anomalistic month because of the varying distance between the earth and the moon, each should be made up of a principal and two minor components, giving the six components in the form

$$y = K_1 \cos (k_1 t + \alpha) + K'_1 \cos [(k_1 + c)\, t + \alpha] + K''_1 \cos [(k_1 - c) t + \alpha]$$
$$+ O_1 \cos (o_1 t + \alpha) + O'_1 \cos [(o_1 + c) t + \alpha] + O''_1 \cos [(o_1 - c) t + \alpha] \quad (41)$$

where c is the speed of the anomalistic month, and the value of α in the various terms is not generally the same.

The speeds of the minor components then have the values:

$$k_1 + c = 15°.041{,}068{,}6 + 0°.544{,}374{,}7 = 15°.585{,}443{,}3$$

$$k_1 - c = 15°.041{,}068{,}6 - 0°.544{,}374{,}7 = 14°.496{,}693{,}9$$

$$o_1 + c = 13°.943{,}035{,}6 + 0°.544{,}374{,}7 = 14°.487{,}410{,}3$$

$$o_1 - c = 13°.943{,}035{,}6 - 0°.544{,}374{,}7 = 13°.398{,}660{,}9$$

The component having the speed $k_1 + c$ is designated as the J_1 component and that having the speed of $o_1 - c$ the Q_1 component. The other two have speeds so close to that of the lunar day, besides being intrinsically small, that they are replaced by a component designated as M_1, with the speed of the lunar day.

69. The components identified in the preceding paragraph are named as follows:

K_1, *lunar portion of the lunisolar diurnal,* whose speed is the sum of those of the lunar day and tropical month.

O_1, *principal lunar diurnal,* whose speed is the difference between those of the lunar day and tropical month.

J_1, *small lunar elliptic,* whose speed is the sum of those of the lunisolar diurnal and the anomalistic month.

Q_1, *larger lunar elliptic,* whose speed is the difference between those of the principal lunar diurnal and of the anomalistic month.

M_1, *smaller lunar elliptic,* whose speed is that of the lunar day.

70. The corresponding solar diurnal components are:

K_1, *solar portion of the lunisolar diurnal,* whose speed is the sum of those of the solar day and the tropical year, or 15°.041,068,6. This unites with the lunar tide of the same speed to form the *lunisolar diurnal component.*

P_1, *principal solar diurnal,* whose speed is the difference between those of the solar day and tropical year, $15-0^\circ.041,068,6=14^\circ.958,931,4$.

The solar elliptic diurnal components corresponding to J_1 and Q_1 are too small to be recognized. Daily land and sea beeezes and daily variations in atmospheric pressure may however give rise to small fluctuations having the period of the mean solar day. A meteorological component, S_1, with a speed of 15° per hour is therefore recognized.

71. *Long period components.*—The following long period components have already been developed in the discussion of the solar and lunar equilibrium tides:

The lunar fortnightly (par. 42).—This component is conventionally represented by the symbol Mf. Its speed is twice that of the tropical month, or 1°.098,033,0 per mean solar hour.

The lunar monthly (par. 43).—Conventionally represented as Mm. Its speed is that of the anomalistic month, or 0°.544,374,7 per mean solar hour.

The solar semiannual (par. 44).—Conventionally represented as Ssa. Its speed is twice that of the tropical year, or 0°.082,137,2 per mean solar hour.

In addition, recurring seasonal meteorological effects produce a solar annual component, designated Sa, whose speed is that of the tropical year, or 0°.041,068,6 per solar hour.

72. *Overtides.*—A distortion of the tides is produced in the comparatively shallow waters of estuaries and other coastal areas. This distortion gives rise to tidal components whose speeds are multiples of the speeds of the astronomical components heretofore developed. They are called overtides because of their analogy to overtones in the theory of musical tones. The only overtides of sufficient magnitude to be of importance are those of the principal lunar and solar components M_2 and S_2. They are designated M_4, M_6, M_8, and S_4 and S_6, the subscripts denoting the ratio of their speeds to that of the mean lunar or solar day.

73. *Compound tides.*—Besides producing overtides, the distortion of the tides in shallow water gives rise to components whose speeds are the sums or differences of the speeds of the elementary components. The recognized compound tides are MS, with a speed of m_2+s_2; MN with a speed of m_2+n_2, MK with a speed of m_2+k_1, 2MK with a speed of m_4-k_1, and 2 SM with a speed of s_4-m_2. These components are generally quite small. Certain other compound tides have the same speed as some of the primary components and therefore unite with them.

74. *Tide depending on the fourth power of the moon's parallax.*—It has been shown in paragraph 32 that the lunar equilibrium tide due to

the fourth power of the moon's parallax is very small, and that it goes through three maxima and minima as the zenith distance of the moon makes a cycle from 0° to 360°; while the principal part of the tide, due to the third power of the moon's parallax, goes through two maxima and minima during this cycle. It is not difficult to see, therefore, that the fluctuations of both the equilibrium and the actual tides due to the fourth power of the moon's parallax go through three fluctuations each lunar day. This tide is therefore approximated by the component:

$$y = \mathrm{M}_3 \cos (\mathrm{m}_3 t + \alpha) \tag{42}$$

where m_3 is three times the speed of the lunar day, or 43°.476,156,3 per mean solar hour. Additional components to account for the variations in this part of the tide due to the changing declination and distance of the moon, and other variations from this primary component, are too small to be recognized.

75. *Résumé of components identified.*—For the purpose of harmonic analysis, it is convenient to group together the components whose speeds are multiples of another. The components identified in the preceding paragraphs fall into groups and individual components as follows:

TABLE II

Symbol	Name	Speed in degrees per mean solar hour
	M GROUP	
M_1	Smaller lunar elliptic, diurnal	14.492,052,1
M_2	Principal lunar semidiurnal	28.984,104,2
M_3	From 4th power moon's parallax	43.476,156,3
M_4	Lunar overtide	57.968,208,4
M_6	do	86.952,312,6
M_8	do	115.936,416,8
	S GROUP	
S_1	Meteorological	15.000,000,0
S_2	Principal solar semidiurnal	30.000,000,0
S_4	Solar overtide	60.000,000,0
S_6	do	90.000,000,0
	K GROUP	
K_1	Lunisolar diurnal	15.041,068,6
K_2	Lunisolar semidiurnal	30.082,137,2
	INDIVIDUAL DIURNAL COMPONENTS	
O_1	Principal lunar diurnal	13.943,035,6
P_1	Principal solar diurnal	14.958,931,4
Q_1	Larger lunar elliptic diurnal	13.398,660,9
J_1	Small lunar elliptic diurnal	15.585,443,3
	INDIVIDUAL SEMIDIURNAL COMPONENTS	
N_2	Larger lunar elliptic semidiurnal	28.439,729,5
L_2	Smaller lunar elliptic semidiurnal	29.528,478,7
T_2	Larger solar elliptic semidiurnal	29.958,933,3
R_2	Smaller solar elliptic semidiurnal	30.041,066,7

TABLE II—Continued

Symbol	Name	Speed in degrees per mean solar hour
	LONG PERIOD COMPONENTS	
Mf	Lunar fortnightly	1.098,033,0
Mm	Lunar monthly	.544,374,7
Ssa	Solar semiannual	.082,137,2
Sa	Solar annual	.041,068,6
	COMPOUND TIDES	
MS		58.984,104,2
MN		57.423,833,7
MK		44.025,172,9
2MK		42.927,139,8
2SM		31.015,895,8

76. *Other components.*—Additional small components disclosed by the mathematical analysis later outlined are here appended for convenient reference.

TABLE III

Symbol	Name	Speed
2N	Lunar elliptic, 2d order, semidiurnal	27.895,354,8
ν_2(nu)	Larger lunar evectional, semidiurnal	28.512,583,1
λ_2(lambda)	Smaller lunar evectional, semidiurnal	29.455,625,3
μ_2(mu)	Variational	27.968,208,4
OO	Lunar diurnal, 2d order	16.139,101,7
2Q	Lunar elliptic, 2d order, diurnal	12.854,286,2
ρ_1(rho)	Larger lunar evectional, diurnal	13.471,514,5
MSf	Lunisolar synodic, fortnightly	1.015,895,8

HARMONIC ANALYSIS OF TIDES

77. The *amplitude* and *initial phase* of each component of the tide at any tidal station may be computed from the observed hourly tidal heights for a sufficient period of days, and the predetermined speed of the component, by the process of *harmonic analysis*, which will now be explained. The observed heights used for this computation are taken at (mean solar) hourly intervals, beginning at midnight (0 hour) each day, giving 24 observations per day. The observations ordinarily are on standard time; but early records may be on local time.

78. *Separation of* S *group of components.*—The repeating form of the sinusoidal curve representing any component (fig. 16) shows at once that the value of the component at any instant is repeated at the intervals of time given by the *period* of the component, and by any multiple of that period. Thus since the period of the S_2 component is $360°/30° = 12$ hours, this component has exactly the same value at say 3 p. m. on any day as at 3 a. m.; and has the same value at 3 a. m. on every succeeding day. The solar overtides S_4 and S_6, with periods of 6 hours and 4 hours respectively, each have similarly the same value at the same hour each succeeding day, as has the small S_1 component. All other components have values which progressively vary at the

same solar hour each succeeding day. The progressive variation of the M_2 component, whose period is 12.420 hours, is illustrated for example, in figure 23, in which P_1 P_2, and P_3, etc., show the relative values of the component at 3 a. m. on 3 successive days; the amplitude and initial phase of the component being taken at random. In this

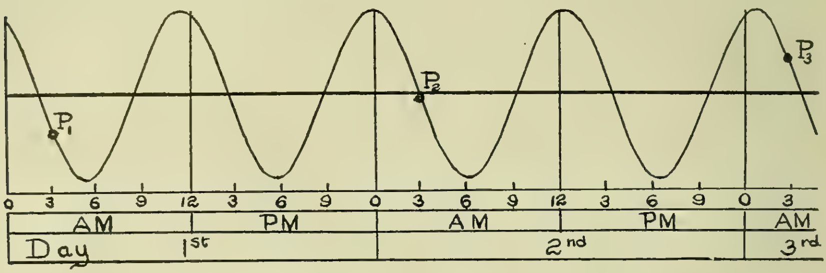

FIGURE 23.—Variation in M_2 component at given solar hour.

progressive variation, these components run through their entire range of values, both positive and negative.

While, therefore, the sum of the successive values of the S group of components at a given hour over a period of days increases directly with the number of days in the period, the average value remaining constant, the sum of all other components at that hour does not so increase. On the contrary, at the intervals at which the positive and negative values of a component offset each other, the sum of the values of that component nearly disappears. If the number of days in the period is suitably chosen, the average value of the resultant of all of the components at a given hour reduces therefore to nearly the value of the S group at that hour.

79. The observed hourly tidal heights are ordinarily scaled from the record of a recording tide gage, and give these heights above an arbitrarily chosen datum, usually set low enough to make all of the readings positive. Each of the recorded heights is then the algebraic sum of the height of mean sea level above datum, plus the resultant of all the tidal components at the hour, plus the accidental variations due to meteorological disturbances, as well as to inaccuracies of observation. The average of the heights at a given hour of the day over a suitably chosen number of days, then closely approximates the height of the resultant of the S group of components at that hour above the datum plane, the other components, together with the accidental variations, being averaged out by the process.

80. *Example.*—The period of the diurnal component K_1 is so close to that of the S_1 component that 6 months of observations are necessary to segregate the S group of components as a whole. The principal solar component, S_2 (with the overtides) may however be approximately determined from a set of observations extending over 15 days,

by averaging the hourly tidal heights at the 12-hour period of this component. For example, the hourly tidal heights at Sitka, Alaska, for the first 3 days, and the average of both the morning and afternoon hourly tides for a 15-day period beginning July 1, 1893, are shown below, the heights during the other days of the period being omitted for brevity.

Hourly height of tide, in feet

Day	1		2		3		Total 3 days	Total 15 days	Average
Hour	*a. m.*	*p. m.*	*a. m.*	*p. m.*	*a. m.*	*p. m.*			
0	13.9	9.5	13.1	8.4	11.6	7.1	63.6	309.2	10.31
1	14.5	11.4	14.1	10.5	13.0	9.2	72.7	317.9	10.60
2	14.2	12.5	14.4	12.1	13.8	11.2	78.2	319.5	10.65
3	13.0	12.8	13.8	12.9	13.8	12.4	78.7	313.5	10.45
4	10.9	12.3	12.2	12.8	12.9	12.8	73.9	301.4	10.05
5	8.5	11.1	10.1	12.0	11.2	12.4	65.3	287.0	9.57
6	6.0	9.8	7.5	10.7	8.9	11.3	54.2	272.9	9.09
7	4.3	8.8	5.5	9.4	6.8	10.0	44.8	264.6	8.82
8	3.5	8.4	4.1	8.5	4.9	8.8	38.2	262.0	8.73
9	3.9	8.8	3.9	8.3	4.1	7.9	36.9	268.0	8.93
10	5.3	10.0	4.7	8.9	4.2	8.0	41.1	280.5	9.35
11	7.5	11.5	6.5	10.2	5.3	8.7	49.7	296.2	9.87
Average									9.70

The average heights derived in the last column are then the approximate hourly heights of the S_2 component, and its overtides,

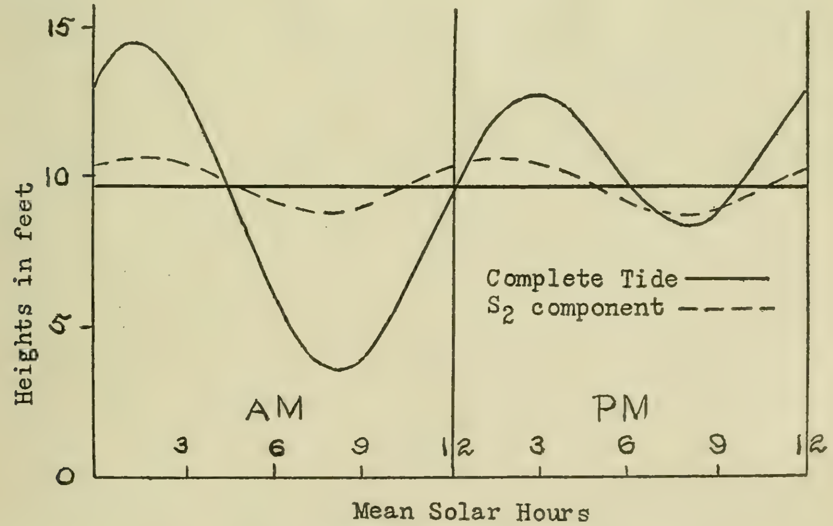

FIGURE 24.—S_2 component at Sitka, Alaska.

above datum. The recorded hourly heights for the first day, and of the S_2 component as thus computed, are plotted in figure 24.

81. The amplitude of the S_2 component at Sitka shown by the plotted curve conforms well with value of 1.137 feet found from a year's observations. The symmetry of the curve indicates that this com-

ponent is not affected by overtides of any substantial amplitude. A complete analysis for a year confirms this fact. This symmetry shows also that a summation for even this short period nearly eliminates the other components.

82. The computation of the S_2 component in paragraph 80 is arranged to illustrate the process most directly. In the conventional form of computation, the observed hourly heights, numbered from 0 to 23 each day, are entered in daily vertical columns on standard sheets. The S group of components is computed by summing the observed heights horizontally, the aggregates of these sums being checked against the aggregate of the sums of the daily columns. The amplitude and the initial phase of each component of the S group are then computed from the average hourly heights of the whole set of observations by a method to be explained later.

83. *Separation of other components.*—The observed tidal heights at each mean *lunar* hour similarly could be taken off and tabulated by *lunar* days, and the M group of components segregated by averaging the tidal heights at each hour of the lunar day. In like manner all of the components could be determined by averaging the tidal heights at their *component hours*, tabulated by *component days.* The process of taking off new observed tidal heights for each of the several components would be a laborious one. Instead, the height at each lunar or other component hour is taken as the observed height at the nearest mean solar hour. These heights are sometimes a little greater and sometimes a little less than the true heights at the component hour, but their average over a sufficient number of days closely approximates the average at the given component hour. The same process of averaging which separates the component sought from the others, reduces the observed heights as taken at the nearest mean solar hour to the heights at the component hour. The correction for a small systematic error resulting from the process is developed in paragraph 97.

84. *Component days and hours.*—To select the observed mean solar hourly heights which are to be taken as the component hourly heights, a tabulation is prepared showing the component hour nearest to each solar hour on each successive calendar day. For diurnal components, the *component day* is the period of the component, and the *component hour,* one twenty-fourth part of that period. For semidiurnal components, the component day is twice the period of the component, and the component hour is one-twelfth of the period. For components of shorter periods, the component day is that multiple of the period nearest to a mean solar day. It will be observed that the components M_1, M_2, M_3, M_4, M_6, and M_8 all have the same component hour; and that the components K_1 and K_2 similarly have the same

component hour. Since the period, in mean solar hours, of a component whose speed is a, is $360°/a$, the length of the component hour of a diurnal component is $360/24a=15/a$ solar hours, and 1 solar hour is $a/15$ component hours. For semidiurnal components, 1 solar hour is $a/30$ component hours.

85. *Tabulation of component hours.*—A tabulation of the mean lunar hours corresponding to mean solar hours will illustrate the process of preparing such a tabulation for any component. Since the speed of the lunar day (and of the M_1 component) is 14°.492,052,1 per hour, 1 solar hour is equal to 14.492,052,1/15=0.966,136,8 lunar hours. In the tabulation at the end of this paragraph, the left-hand column lists the mean solar hours each day. The next column gives, in parentheses, the corresponding lunar hours, to three places of decimals, for the first 15 hours of the first calendar day, and the third column, the corresponding nearest whole lunar hour. Succeeding columns give these whole lunar hours on the following calendar days. It will be observed that the values of the corresponding lunar hours diminish hourly by 1.−0.966,136,8=0.033,863,2. Between the fourteenth and fifteenth mean solar hours the cumulative diminution passes half a unit, so that in whole numbers the fourteenth lunar hour corresponds to both the fourteenth and fifteenth mean solar hours. Obviously from the fifteenth solar hour on, until the progressive diminution passes 1.5, the corresponding lunar hours are 1 hour less than the solar hours. After 1.5/0.033,863,2=44.296 hours, i. e., beginning with the twenty-first hour of the second day, the corresponding lunar hours drop back another unit, and may be found by subtracting 2 from the solar hour (or adding 22 if the remainder would be negative), and so on. To fill out the tabulation it is necessary only to find the solar hours at which the lunar hours drop back a unit. These occur at intervals of 1/0.033,863,2=29.53058 hours beginning with 0.5/0.033,863,2=14.76529 hours. They are, therefore, the integers following:

14.76529, or 1st day—15 hours.	(−1)
44.29586, or 2d day—21 hours.	(−2)
73.82644, or 4th day— 2 hours.	(−3)

and so on.

The hours beginning at which successive numbers are to be added or subtracted to give the nearest component hour of each of the harmonic components are tabulated in Manuals of Harmonic Analysis of Tides under the heading "Tables for the Construction of Primary Stencils;" from which the data for the M group of components, up to the twenty-ninth day, is extracted.

M *components*

Differences		Day	Hour	Differences		Day	Hour	Differences		Day	Hour
0		1	0	+16	−8	10	6	+8	−16	20	2
+23	−1	1	15	+15	−9	11	12	+7	−17	21	8
+22	−2	2	21	+14	−10	12	17	+6	−18	22	13
+21	−3	4	2	+13	−11	13	23	+5	−19	23	19
+20	−4	5	8	+12	−12	15	4	+4	−20	25	0
+19	−5	6	13	+11	−13	16	10	+3	−21	26	6
+18	−6	7	19	+10	−14	17	15	+2	−22	27	11
+17	−7	9	0	+9	−15	18	21	+1	−23	28	17
								0		29	22

It is for the preparation of such tables for 369 or more days that the speeds of the components are extended to seven places of decimals.

The application of these tables is illustrated in the extension of the subjoined tabulation to include the seventh day.

Mean solar hour	Mean lunar hour	Calendar days 1	2	3	4	5	6	7	Mean solar hour	Mean lunar hour	Calendar days 1	2	3	4	5	6	7
		Nearest whole lunar hour									Nearest whole lunar hour						
0	(0)	0	23	22	22	21	20	19	12	(11.593)	12	11	10	9	8	8	7
1	(0.966)	1	0	23	23	22	21	20	13	(12.559)	13	12	11	10	9	8	8
2	(1.932)	2	1	0	23	23	22	21	14	(13.525)	14	13	12	11	10	9	9
3	(2.898)	3	2	1	0	0	23	22	15	(14.491)	14	14	13	12	11	10	10
4	(3.864)	4	3	2	1	1	0	23	16		15	15	14	13	12	11	11
5	(4.831)	5	4	3	2	2	1	0	17		16	16	15	14	13	12	12
6	(5.797)	6	5	4	3	3	2	1	18		17	17	16	15	14	13	13
7	(6.763)	7	6	5	4	4	3	2	19		18	18	17	16	15	14	13
8	(7.729)	8	7	6	5	4	4	3	20		19	19	18	17	16	15	14
9	(8.693)	9	8	7	6	5	5	4	21		20	19	19	18	17	16	15
10	(9.661)	10	9	8	7	6	6	5	22		21	20	20	19	18	17	16
11	(10.627)	11	10	9	8	7	7	6	23		22	21	21	20	19	18	17

The height of the M group of components at zero component hour is found by summing and averaging the observed heights at the hours marked 0 on the tabulation; the height at the first component hour by averaging those marked 1, and so on for the 24 component hours.

86. *Example.*—The hourly heights of the M components found from the application of this process to the observed heights at Sitka, Alaska, over a period of 29 days beginning July 1, 1893, are as follows:

Component hour	0	1	2	3	4	5	6	7	8	9	10	11
Sums	334. 9	370. 4	369. 9	371. 8	345. 4	277. 7	239. 9	203. 8	187. 3	197. 0	233. 6	272. 3
Divisors	29	29	28	29	30	28	29	29	29	29	29	28
Component height	11. 55	12. 77	13. 21	12. 82	11. 51	9. 92	8. 27	7. 03	6. 46	6. 79	8. 06	9. 72

Component hour	12	13	14	15	16	17	18	19	20	21	22	23
Sums	334. 7	382. 9	386. 2	373. 8	341. 3	289. 6	241. 8	213. 6	182. 2	197. 6	244. 4	288. 8
Divisors	29	30	29	29	29	29	29	30	28	29	30	29
Component height	11. 54	12. 76	13. 32	12. 89	11. 77	9. 99	8. 34	7. 12	6. 51	6. 81	8. 15	9. 96

The heights of the M components thus computed, and for comparison the recorded heights for the first day's observations are shown in figure 25.

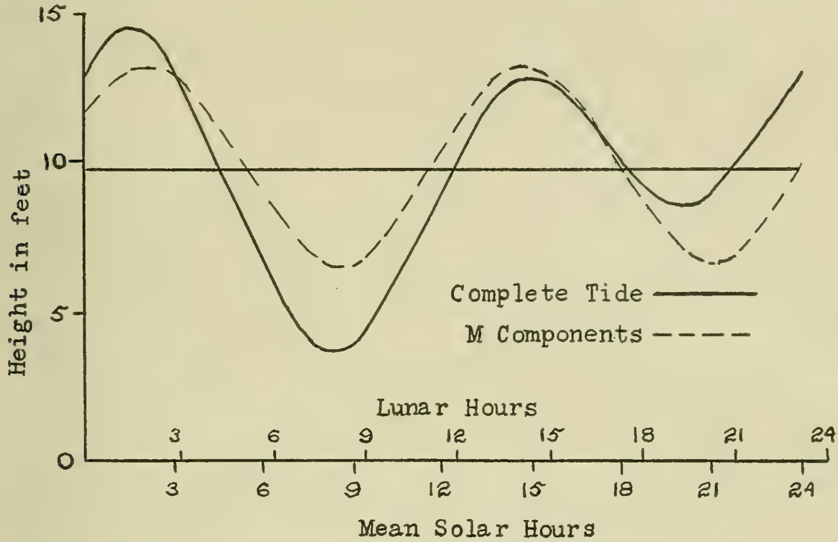

FIGURE 25.—*M* components at Sitkr, Alrska.

87. *Stencils.*—The summation of the hourly heights is facilitated by the ingenious device of cutting stencils which, when laid over the tabulated observed heights on a sheet in standard form, show through the openings the heights to be added to give the sums for each component hour. Two stencils are prepared for each successive 7-day period shown on a standard sheet, one for the even component hours and one for the odd, lines being drawn on the stencil to connect the observations to be taken for each component hour.

88. The stencils ordinarily used are prepared for the program of computation illustrated for the M component, in which each observed height enters once and only once in the summations of the 24 component hours. On the stencils for those components whose component hour, like the lunar hour, is longer than the mean solar hour, a component hour is repeated at the intervals at which its tabulation must shift back a unit to most nearly correspond to the mean solar hour; and both of the observed values so indicated are included in the summation for the component hour. If the component hour is shorter than a mean solar hour, a component hour is omitted from the stencil at the corresponding intervals at which the tabular values must shift forward a unit. The aggregate of the sums for the 24 component hours taken from stencils in this form may be checked against the sum of all of the observations for the period, but the divisors for computing the averages may not be the same as the number of component

days in the period. Stencils may be prepared in which the summations are made for 1 and only 1 component hour in each component day. In such stencils 1 of the observed heights is omitted at the shifts of the component hour when it is longer than the mean solar hour, and 2 component hours are assigned to the same observed height at the shifts when the component hour is shorter than the mean solar. The latter system is not generally favored.

89. *Secondary stencils.*—The speed of the K_1 component is so nearly that of the solar day that its component hours shift with respect to the solar hours but once in about 8 days. To facilitate the computation of this component from a year's observations, it is permissible to assemble on standard sheets the 7-day sums of the observed hourly heights, and to use these sums, instead of the daily observations, in computing the hourly heights of this component. A number of components similarly have speeds so close to those of others that they may be computed from the 7-day hourly sums of their primary component. The stencils prepared for such computations are called secondary stencils.

90. *Number of days of observations required.*—To eliminate a component by harmonic analysis, the tidal observations should extend over a period, or multiple thereof, in which the successive values of the component to be eliminated, at the component hours of the component sought, run through their entire range of values, both positive and negative (par. 78). If a is the speed of the component A, which is to be segregated, and b the speed of a component B which is to be eliminated, component B gains $b-a$ degrees on component A each solar hour. Its successive daily values at any component hour of A will then run through their whole range of values in $360°/24\,(b-a) = 15/(b-a)$ days, or $360/(b-a)$ hours, the synodic period of the two components (par. 54). A consideration of the relative speeds of the tidal components establishes a minimum period of 14 days for diurnal and 15 days for semidiurnal components. The periods adopted by the United States Coast and Geodetic Survey are 14–15, 29, 58, 87, 105, 134, 163, 192, 221, 250, 279, 297, 326, 355, and 369 days. The standard period for a complete analysis for tide predicting purposes is 369 days. A period of 29 days affords, however, fair determinations if corrections are applied to eliminate the residual effect of interfering components.

91. *Computation of amplitude and initial phase.*—The determination of the heights of the various components at their component hours has been described in the preceding paragraphs. The amplitude and initial

phase of each component are computed from its hourly heights by a process based on the arithmetical integration of the coefficients of Fourier's series. Components having the same component hour are separated by the process.

92. Using the form indicated in equation (29), the height, at any time t, of the resultant of a group of components having the same component hour is given by the equation:

$$y=H_0+A_1 \cos (at-\zeta_1)+A_2 \cos (2at-\zeta_2)+A_3 \cos (3at-\zeta_3)+ \ldots \quad (43)$$

where H_0 is the height of mean sea level above datum, A_1 the amplitude of the diurnal component, a its speed, and ζ/a the time of its high water (par. 49); A_2 is the amplitude of the semidiurnal component and $\zeta/2a$ the time of its high water; and the other terms similarly represent minor components and overtides. But a few terms are needed in any group of components having the same component hour, and for single components the equation reduces to one variable term in addition to the constant term H_0.

The expansion of the cosines in equation (43) gives the equation:

$$\begin{aligned} y=&H_0+A_1 \cos at \cos \zeta_1+A_1 \sin at \sin \zeta_1+A_2 \cos 2at \cos \zeta_2 \\ &+A_2 \sin 2at \sin \zeta_2+A_3 \cos 3at \cos \zeta_3+A_3 \sin 3at \sin \zeta_3 \ldots \quad (44) \end{aligned}$$

Placing:

$$A_1 \cos \zeta_1=c_1,\ A \sin \zeta_1=s_1,\ A_2 \cos \zeta_2=c_2,\ A_2 \sin \zeta_2=s_2, \text{ etc.} \quad (45)$$

equation 44 becomes:

$$\begin{aligned} y=&H_0+c_1 \cos at+s_1 \sin at+c_2 \cos 2at+s_2 \sin 2at+c_3 \cos 3at \\ &+s_3 \sin 3at+ \ldots \quad (46) \end{aligned}$$

The values of the angles and coefficients in equation (43) may be found readily from the coefficients c_1, s_1, c_2, s_2, c_3, s_3, etc., of equation (46), since, from equations (45):

$$\tan \zeta_1=s_1/c_1 \qquad \tan \zeta_2=s_2/c_2 \qquad \tan \zeta_3=s_3/c_3, \text{ etc.} \quad (47)$$

and:

$$A_1=c_1/\cos \zeta_1=s_1/\sin \zeta_1 \qquad A_2=c_2/\cos \zeta_2=s_2/\sin \zeta_2, \text{ etc.} \quad (48)$$

It may be noted that by expressing equation (43) in the form indicated by equation (29), rather than equation (27), negative signs are avoided in equations (46), (47), and (48).

93. The demonstration of Fourier's series shows that if y is any function of x, its values between the limits of $x=0$ and $x=l$ are given by the series:

$$y=B_0+B_1 \cos(\pi x/l)+C_1 \sin(\pi x/l)+B_2 \cos(2\pi x/l)+C_2 \sin(2\pi x/l) \\ +B_3 \cos(3\pi x/l)+C_3 \sin(3\pi x/l)+\ .\ .\ . \qquad (49)$$

in which:

$$B_0=(1/l)\int_0^l y dx,$$

$$B_1=(2/l)\int_0^l y \cos(\pi x/l)dx,\quad C_1=(2/l)\int_0 y \sin(\pi x/l)dx,$$

$$B_2=(2/l)\int_0^l y \cos(2\pi x/l)dx,\ C_2=(2/l)\int_0^l y \sin(2\pi x/l)dx,$$

$$B_3=(2/l)\int_0^l y \cos(3\pi x/l)dx,\ C_3=(2/l)\int_0^l y \sin(3\pi x/l)dx, \qquad (50)$$

and so on. The angles in these equations are expressed in radians.

94. If T is the length (in mean solar hours) of the component day, then since 2π radians$=360°$, $2\pi/T=a$, the speed of the diurnal component.

Placing $x=t$ and $l=T$, equation (49) becomes:

$$y=B_0+B_1 \cos \tfrac{1}{2}at+C_1 \sin \tfrac{1}{2}at+B_2 \cos at+C_2 \sin at+B_3 \cos 3/2\ at \\ +C_3 \sin 3/2\ at+B_4 \cos 2at+C_4 \sin 2at+\ .\ .\ . \qquad (51)$$

Equation (51) is the development of *any* function of t. It becomes the development of the particular function of t expressed by equation (46) if it is identical with the latter, i. e., if the coefficients of the identical terms in the two equations are equal, the coefficients of the terms in equation (51) not appearing in equation (46) becoming zero. It follows therefore that

$$H_0=B_0=(1/T)\int_0^T y dt,$$
$$c_1=B_2=(2/T)\int_0^T y \cos at\ dt,\quad s_1=C_2=\int_0^T y \sin at\ dt,$$
$$c_2=B_4=(2/T)\int_0^T y \cos 2\ at\ dt,\ s_2=C_4=\int_0^T y \sin 2\ at\ dt, \text{ etc.} \qquad (52)$$

An examination of the form of the integrals in equations (52) discloses that $(1/T)\int_0^T ydt$ is the mean value of y between the limits of

0 and T; $(2/T)\int_0^T y \cos at\, dt$ is twice the mean value of $y \cos at$ between the same limits; and that similarly all of the remaining values of the coefficients are twice the mean values of the expressions integrated. If then the values of y, or the determined heights of the component at its successive component hours, are designated as h_0, h_1, $h_2 \cdots h_{23}$, it immediately follows that:

$$H_0 = \tfrac{1}{24}(h_0 + h_1 + h_2 + \ \dots\ + h_{23}) \tag{53}$$

The angle at has the value of $\frac{360°}{24} = 15°$ at the end of the first component hour, 30° at the end of the second component hour, etc. Twice the average values of $y \cos at$; $y \sin at$; $y \cos 2at$; $y \sin 2at$ are then:

$$c_1 = \tfrac{1}{12}(h_0 \cos 0 + h_1 \cos 15° + h_2 \cos 30° + . + h_{23} \cos 345°) \tag{54}$$
$$s_1 = \tfrac{1}{12}(h_0 \sin 0 + h_1 \sin 15° + h_2 \sin 30° + . + h_{23} \sin 345°) \tag{55}$$
$$c_2 = \tfrac{1}{12}(h_0 \cos 0 + h_1 \cos 30° + h_2 \cos 60° + . + h_{23} \cos 330°) \tag{56}$$
$$s_2 = \tfrac{1}{12}(h_0 \sin 0 + h_1 \sin 30° + h_2 \sin 60° + . + h_{23} \sin 330°) \tag{57}$$

and so on.

Equation (53) merely expresses the evident fact that the elevation of mean sea level is the mean of the heights at the component hours. The amplitude and initial phase of the diurnal component are determined by computing c_1 and s_1 from equations (54) and (55) and applying to them the relations expressed by equations (47) and (48). The amplitude and initial phase of the semidiurnal component are similarly derived from the computed values of c_2 and s_2; and those of other components from the corresponding coefficients, the equations for determining which may be written by analogy to equations (54) to (57).

The computations of the coefficients from equations (54) to (57) may be greatly abbreviated by combining the terms whose sine or cosine factors have the same numerical value. For example, in finding the values of c_2 and s_2 from equations (56) and (57), it is apparent that these factors for h_{12} are respectively cos 360° and sin 360°, those for h_{13} are cos (360°+30°) and sin (360°+30°) and so on. The successive factors for the last 12 terms are consequently the same as for the first 12 terms. Furthermore, the factors for h_6 are cos 180° and sin 180°, respectively. Since $\cos(180+\phi) = -\cos\phi$ and $\sin(180°+\phi) = -\sin\phi$, the successive factors for the second 6 terms are equal but opposite in sign to those of the first 6 terms.

95. *Example.*—Taking the heights of the M group of components at their component hours at Sitka, computed in paragraph 86, the computation of the amplitude and initial phase of the M_2 component

at this station and at the period of the observations may be arranged as follows:

(1) Hourly heights h_0-h_{11}	11.55	12.77	13.21	12.82	11.51	9.92	8.27	7.03	6.46	6.79	8.06	9.72
(2) Hourly heights $h_{12}-h_{23}$	11.54	12.76	13.32	12.89	11.77	9.99	8.34	7.12	6.51	6.81	8.15	9.96
(3) Sums	23.09	25.53	26.53	25.71	23.28	19.91	16.61	14.15	12.97	13.60	16.21	19.68
(4) Sums h_6 to h_{11}	16.61	14.15	12.97	13.60	16.21	19.68						
(5) Differences	6.48	11.38	13.56	12.11	7.07	.23						

1	2	3	4		5	6
Combined heights	Angles	Cosines	Products (1×3)		Sines	Products (1×5)
6.48	0	1	6.48		0	0
11.38	30°	.866	9.855		.5	5.690
13.56	60°	.5	6.780		.866	11.743
12.11	90°	0			1	12.110
7.07	120°	−.5		−3.535	.866	6.123
.23	150°	−.866		−.199	.5	.115
			23.115	−3.734		35.781
			−3.734			
			$12c_2$=19.381			$12s_2$=35.781
			c_2= 1.6151			s_2= 2.9818

$\log s_2$ = 0.47448
$\log c_2$ = .20820

$\log \tan \zeta$ = .26628
ζ = 61°.6
$\log s_2$ = .47448
$\log \sin \zeta$ = 9.94414

$\log M_2$ = .53034
M_2 = 3.391

$$H_0=\frac{237.27}{24}=9.886$$

The expression for the M_2 component is therefore:

$$y=3.391 \cos (m_2t-61.6°)$$

Figure 26 shows the graph of the M_2 component and the plotted heights (referred to mean sea level) of the resultant of all of the M components as derived. It is apparent that at this station the other

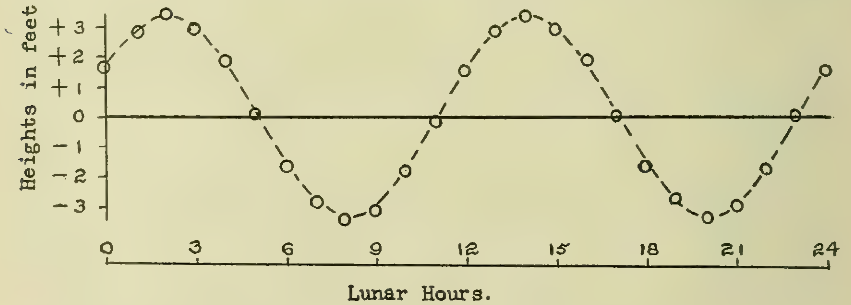

FIGURE 26.—M_2 component and hourly values of M group, Sitka, Alaska.

components of the group are small and that the summation has nearly effected the elimination of components outside of the M group.

It is of interest to note that in the computation as set forth, the summation of lines (1) and (2) automatically eliminates the M_1 and M_3 components, and the subtraction of line (5) from line (4) eliminates the constant height of mean sea level and the M_4 and M_8 components.

96. A reference to the list of components in paragraph 75 shows

that the complete analysis of the M group of components at a station includes the separation of six components. Computation programs for this purpose are shown on standard forms of the United States Coast and Geodetic Survey. The analysis of the S group of components includes the separation of four components; and of the K group 2. All other components require but a single analysis.

97. *Augmenting factors.*—In the computations of the components (except those of the S group) the tidal height at each component hour is taken as the average of the observed heights at the nearest mean solar hour (par. 83). When the computation is made from stencils in the form ordinarily used (par. 88), these observed heights are scattered quite uniformly over an interval extending from one-half a component hour before to one-half a component hour after the exact component hour.

It is graphically apparent from figure 27 that on a sinusoidal curve the average of these heights is always somewhat less, numerically, than the height at the middle of the period, and that a small systematic error is introduced by using the average value. This error is readily corrected, since on a cosine curve the mid height has a constant ratio to the mean height over an arc of given length. This ratio is called the *augmenting factor*.

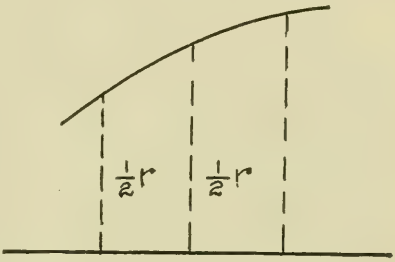

FIGURE 27.

98. To determine the augmenting factor for a component whose equation is $y=A\cos(at+\alpha)$, let r be the length of the componen thour. The average value of y, between the limits of $t_0-\frac{1}{2}r$ and $t_0+\frac{1}{2}r$ is then:

$$(1/r)\int_{t_0-\frac{1}{2}r}^{t_0+\frac{1}{2}r} A\cos(at+\alpha)dt$$

$$\begin{aligned}&=(A/ar)[\sin(at_0+\tfrac{1}{2}ar+\alpha)-\sin(at_0-\tfrac{1}{2}ar+\alpha)]\\&=(A/ar)[\sin(at_0+\alpha+\tfrac{1}{2}ar)-\sin(at_0+\alpha-\tfrac{1}{2}ar)]\\&=2(A/ar)\cos(at_0+\alpha)\sin\tfrac{1}{2}ar\end{aligned} \qquad (58)$$

The ratio of the mid value to the mean value is then:

$$A\cos(at_0+\alpha)/(2A/ar)\cos(at_0+\alpha)\sin\tfrac{1}{2}ar=ar/2\sin\tfrac{1}{2}ar \qquad (59)$$

in which ar is an angle expressed in radians, whose equivalent, in degrees, is $\pi ar/180°$. The expression for the augmenting factor is therefore:

$$\pi ar/360° \sin\tfrac{1}{2}ar$$

For diurnal components, the length of the component hour, r, is $15°/a$, for semidiurnal components $30°/a$, and so on. The values of the augmenting factor are then:

Diurnal components,	$\dfrac{\pi}{24 \sin 7°30'}=1.00286$
Semidiurnal components,	$\dfrac{2\pi}{24 \sin 15°}=1.01152$
M_3,	$\dfrac{3\pi}{24 \sin 22°30'}=1.02617$
M_4,	$\dfrac{4\pi}{24 \sin 30'}=1.04720$
M_6,	$\dfrac{6\pi}{24 \sin 45°}=1.11072$
M_8,	$\dfrac{8\pi}{24 \sin 60°}=1.20920$

99. A review of the process by which the amplitude and initial phase of each component are found (par. 94), shows that the application of the augmenting factor to the hourly component heights (above mean sea level) will increase the amplitude in the same ratio, but will not affect the initial phase. The augmenting factor is therefore applied directly to the computed amplitude. The application of this correction to the amplitude of the M_2 component at Sitka, for example, gives a corrected value of $3.391\times1.01152=3.430$ feet. Evidently, no augmenting factor should be applied to the S components. The more complicated factors for computations from secondary stencils are given in the Manual of Harmonic Analysis of Tides.

100. *Elimination.*—The hourly component heights derived from the process of averaging that has been described will contain the residuals of components other than that sought. After a first determination has been made of the amplitudes and initial phases of the several components, corrections may be computed from them to eliminate from each the effect of the other components. The process is explained in the Manual of Tides, but is not of sufficient general interest to be here included.

101. *Long period components.*—The components listed in paragraphs 75 and 76 include 2 having a fortnightly, 1 a monthly, 1 a semiannual, and 1 an annual period. The first 3 of these are too small to be of much importance, but periodic meteorological causes may produce substantial annual and semiannual variations in the sea level. Since a long period component does not change appreciably during a calendar day, the daily averages of the observed tidal heights, instead of the hourly heights, may be used for its determination, or the daily sums may as well be used, the final result being divided by

24. The computation of the amplitude and initial phase follows the general method heretofore described for the diurnal, semidiurnal, and short-period components. For the fortnightly and monthly components, a *component month* replaces the *component day*. It is divided into 24 parts, corresponding to the component hours. A prepared tabulation designates the daily sums to be taken as the height at each component "hour." These heights are then summed and averaged, and the amplitude and phase of the component computed therefrom. For the annual and semiannual components, the component year similarly replaces the component day. Since the average of the observed heights during a calendar day contains residuals of the short-period components (other than the S components) the amplitudes and phases of the long-period components are corrected by the process of elimination heretofore referred to.

MEAN VALUES AND EPOCHS OF COMPONENTS

102. *Mean values.*—The amplitude of each component of the actual tide increases and decreases with the changing inclination of the moon's orbit to the plane of the earth's Equator, and the amplitudes computed from a particular set of observations must therefore be corrected before they may be used at another period. The correction is based on the logical assumption that the change in the actual components is proportional to the change in the corresponding equilibrium components. For convenience, the amplitude of each component of the actual tide is reduced to its mean value, which is obviously independent of the period from which it was derived.

103. *Epochs.*—The computed initial phase of each component is that at the beginning of the particular set of observations from which the component was derived. The phase of a component of the actual tide depends upon the accidental configuration of the sea bed, while that of the corresponding component of the equilibrium tide is dependent upon astronomical causes alone. Since both components have the same speed, the difference in their phases, at any tidal station, is constant at all times. This difference is called the *epoch* of the component and is conventionally represented by the general symbol κ (kappa). It may readily be found by taking the difference between the initial phase of the equilibrium component, at the zero hour of the observations (as determined from astronomical data) and the initial phase of the actual component, as determined from the observations. The phase of the actual component at any other origin of time can then be found by applying its epoch to the phase of the equilibrium tide at that time. The epoch of a particular component is designated by the symbol for its amplitude with a degree mark. Thus the epoch of the M_2 component is designated as $M_2°$.

104. *Mathematical derivation of lunar equilibrium tide.*—To derive the formulae for reducing the components of the actual tide to their mean values, for converting these mean values to the amplitudes applicable at any given period, and for determining the epochs of the components, it becomes necessary to develop the mathematical expressions for each component of the equilibrium tide in terms of astronomical constants. The expressions for the lunar and solar equilibrium tides in terms of the hour angle and declination of the moon and sun, derived in equation (24), afforded the means for developing the characteristics of the tide, and for inferring therefrom the speeds of most of the components. The much more elaborate expression necessary to develop the coefficients and phase relations of the components of the lunar and solar equilibrium tides will now be developed in outline.

105. In figure 28, N is the north pole of the earth's axis on the celestial sphere, $UIS_1M_1P_1$ the celestial Equator, UOS the ecliptic (the path of the sun), U the vernal equinox and S the position of the mean sun at a given instant; IOM the moon's path (orbit), I its intersection with the Equator, O the moon's node, M the position of the moon, P the zenith of a tidal station, NPP_1 its celestial meridian, NMM_1 the hour circle of the moon, NSS_1 the hour circle of the mean sun, and U_1 the foot of the great circle drawn through the vernal equinox perpendicular to the moon's orbit.

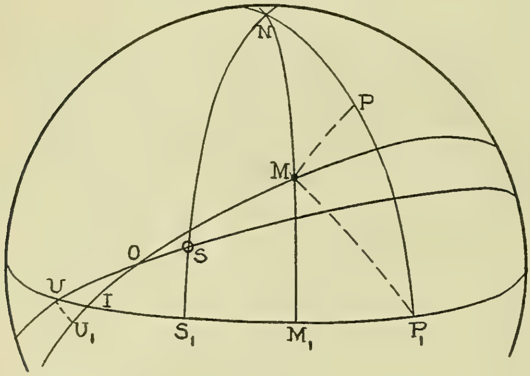

FIGURE 28.

Then:

θ (theta), the arc PM, is the zenith distance of the moon.

δ (delta), the arc M_1M, the declination of the moon.

λ (lamda), the arc P_1P, the latitude of the station.

H, the angle PNM, the hour angle of the moon.

N, the arc UO, the longitude of the moon's node.

I, the angle M_1IM, the inclination of the moon's orbit.

The symbols conventionally assigned to other arcs and angles, and to pertinent astronomical constants are:

T, the arc P_1S_1, the hour angle of the mean sun.

$s+k$, the arc U_1M, the true longitude of the moon.

s, the mean longitude of the moon; i. e., the longitude which it would have it if travelled at the average rate along its orbit.

k, the correction to be added to the mean longitude of the moon to give its true longitude.
h, the arc US_1, the mean longitude of the sun.
p, the mean longitude of lunar perigee, the arc measured from U_1 to the position of lunar perigee, if the latter moved at its mean rate.
ξ (xi), the arc U_1I, the longitude, in the moon's orbit, of the Intersection.
ν (nu), the arc UI, the right ascension of the Intersection.
$e=0.05490$, the eccentricity of the moon's orbit.
$m=0.074804$, the ratio of the mean motion of the sun to that of the moon.
R, the true distance from the center of the earth to the center of the moon at a given moment.
$c=238{,}857$ statute miles, the mean distance, earth to moon.
$a=3{,}958.89$ statute miles, the mean radius of the earth.

The values of e and m given are those for January 1, 1900, but they change but little with the time.

106. The height of the lunar equilibrium tide is, from equation (16):

$$u=\tfrac{1}{2}(Ma^3/ER^3)a(3\cos^2\theta-1) \tag{16}$$

In which M is the mass of the moon and E the mass of the earth. The ratio M/E has a value of 1/81.45.
As shown in equation (21):

$$\cos\theta=\sin\lambda\sin\delta+\cos\lambda\cos\delta\cos H \tag{21}$$

From the right spherical triangle IM_1M:

$$\sin\delta=\sin I\sin IM \tag{60}$$

From the right spherical triangle MM_1P_1

$$\cos\delta\cos H=\cos P_1M \tag{61}$$

And from the spherical triangle IP_1M:

$$\cos P_1M=\cos IM\cos IP_1+\sin IM\sin IP_1\cos I \tag{62}$$

From the figure:

$$IM=U_1M-U_1I=s+k-\xi \tag{63}$$

and:

$$IP_1=US_1+S_1P_1-UI=h+T-\nu \tag{64}$$

107. By substituting, in equation (21), the expressions for sin δ, and cos δ cos H, derived from equations (60) to (64), an expression for cos θ may be derived in terms of s, k, h, T, ξ, and ν.

The astronomical formula for the correction k (in radians) is:

$$k=2e \sin (s-p)+5/4\, e^2 \sin 2(s-p)+15/4\, me \sin (s-2h+p)$$
$$+11/8\, m^2 \sin 2\,(s-h) \qquad (65)$$

and since k is small, its angle, in radians, may be substituted for its sine.

The astronomical formula for $1/R$, in equation (16) is

$$1/R=1/c+e \cos (s-p)/c(1-e^2)+e^2 \cos 2(s-p)/c(1-e^2)$$
$$+15/8\, me \cos (s-2h+p)/c(1-e^2)+m^2 \cos 2(s-h)/c(1-e^2) \quad (66)$$

108. The expression for the lunar equilibrium tide in terms of the angles λ, T, s, h, ξ, and ν, and astronomical constants, is then derived by substituting these expressions for cos θ and $1/R$ in equation (16) and successively converting the products of the sines and cosines of the angles T, s, h, ξ, and ν, into sines and cosines of their sums and differences, by the application of the elementary trigonometric formulas:

$$\begin{aligned}
\cos x \cos y&=\tfrac{1}{2} \cos (x-y)+\tfrac{1}{2} \cos (x+y)\\
\sin x \sin y&=\tfrac{1}{2} \cos (x-y)-\tfrac{1}{2} \cos (x+y)\\
\sin x \cos y&=\tfrac{1}{2} \sin (x+y)+\tfrac{1}{2} \sin (x-y)\\
\cos x \sin y&=\tfrac{1}{2} \sin (x+y)-\tfrac{1}{2} \sin (x-y) \qquad (67)\\
\cos^2 x&=\tfrac{1}{2}(1+\cos 2x)\\
\sin^2 x&=\tfrac{1}{2}(1-\cos 2x)\\
\sin x \cos x&=\tfrac{1}{2} \sin 2x
\end{aligned}$$

109. The result is an equation for u which contains 63 terms, and which would cover more than a printed page. It is not here repeated. But 21 of the variable terms have coefficients of sufficient numerical value to require consideration. These give the following working equation for the lunar equilibrium tide, now designated as y:

$$y=3/2\ (Ma^3/Ec^3)a\times$$
$$\begin{aligned}
\{\cos^2 \lambda \cos^4 \tfrac{1}{2}I\ [(\tfrac{1}{2}-5/4\, e^2) &\cos (2T+2h-2s+2\xi-2\nu) && \mathrm{M_2}\\
+7/4\, e &\cos (2T+2h-3s+p+2\xi-2\nu) && \mathrm{N_2}\\
+1/4\, e &\cos (2T+2h-s-p+2\xi-2\nu+180°) && [\mathrm{L_2}]\\
+17/4\, e^2 &\cos (2T+2h-4s+2p+2\xi-2\nu) && \mathrm{2N}\\
+105/32\, me &\cos (2T+4h-3s-p+2\xi-2\nu) && \nu_2\\
+15/32\, me &\cos (2T-s+p+2\xi-2\nu+180°) && \lambda_2
\end{aligned}$$

$$+23/16\ m^2 \cos\ (2T+4h-4s+2\xi-2\nu)] \qquad \mu_2$$
$$+\cos^2 \lambda \sin^2 I\ [(1/4+3/8\ e^2) \cos\ (2T+2h-2\nu) \qquad [\mathrm{K}_2]$$
$$+3/8\ e \cos\ (2T+2h-s+p-2\nu)] \qquad [\mathrm{L}_2]$$
$$+\sin 2\lambda \sin I \cos^2 \tfrac{1}{2}I\ [(1/2-5/4\ e^2) \cos\ (T+h-2s+2\xi-\nu+90°) \qquad \mathrm{O}_1$$
$$+7/4\ e \cos\ (T+h-3s+p+2\xi-\nu+90°) \qquad \mathrm{Q}_1$$
$$+1/4\ e \cos\ (T+h-s-p+2\xi-\nu-90°) \qquad [\mathrm{M}_1]$$
$$+17/4\ e^2 \cos\ (T+h-4s+2p+2\xi-\nu+90°) \qquad 2\mathrm{Q}$$
$$+105/32\ me \cos\ (T+3h-3s-p+2\xi-\nu+90°)] \qquad \rho_1$$
$$+\sin 2\lambda \sin I \sin^2 \tfrac{1}{2}I\ (1/2-5/4\ e^2) \cos\ (T+h+2s-2\xi-\nu-90°) \qquad \mathrm{OO}$$
$$+\sin 2\lambda \sin 2I\ [(1/4+3/8\ e^2) \cos\ (T+h-\nu-90°) \qquad [\mathrm{K}_1]$$
$$+3/8\ e \cos\ (T+h+s-p-\nu-90°) \qquad \mathrm{J}_1$$
$$+3/8\ e \cos\ (T+h-s+p-\nu-90°)] \qquad [\mathrm{M}_1]$$
$$+(1/2-3/2 \sin^2 \lambda) \sin^2 I(1/2-5/4\ e^2) \cos\ (2s-2\xi) \qquad \mathrm{Mf}$$
$$+(1/2-3/2 \sin^2 \lambda)\ (1-3/2 \sin^2 I)\ [e \cos\ (s-p) \qquad \mathrm{Mm}$$
$$+m^2 \cos\ (2s-2h)]\}. \qquad [\mathrm{Msf}]$$
$$(68)$$

110. *Solar equilibrium tide.*—The corresponding equation for the solar equilibrium tide may be written at once from equation (68) by substituting:

S, the mass of the sun, for M the mass of the moon.
c_1, the mean distance of the sun, for c.
e_1, the eccentricity of the sun's orbit, for e.
ω (omega), the obliquity of the ecliptic, for I.
p_1, the longitude of the sun's perigee, for p.

The angle s, the mean longitude of the moon, becomes identical with h, the mean longitude of the sun. The angles ξ and ν, the longitude and right ascension of the intersection, become zero, as does m, the relative motion of the moon and the sun; and e_1 is so small that a number of terms dependent upon this constant may be dropped. The equation of the solar equilibrium tide then becomes:

$$y=3/2\ (Sa^3/Ec_1^3)a\{\cos^2 \lambda \cos^4 \tfrac{1}{2}\omega\ [(1/2-5/4\ e_1^2) \cos 2T \qquad \mathrm{S}_2$$
$$+7/4\ e_1 \cos\ (2T-h+p_1) \qquad \mathrm{T}_2$$
$$+1/4\ e_1 \cos\ (2T+h-p_1+180°)] \qquad \mathrm{R}_2$$
$$+\cos^2 \lambda \sin^2 \omega(1/4+3/8\ e_1^2) \cos\ (2T+2h) \qquad [\mathrm{K}_2]$$
$$+\sin 2\lambda \sin \cos^2 \tfrac{1}{2}\omega(1/2-5/4\ e_1^2) \cos\ (T-h+90°) \qquad \mathrm{P}_1$$
$$+\sin 2\lambda \sin 2\omega(1/4+3/8\ e_1^2) \cos\ (T+h-90°) \qquad [\mathrm{K}_1]$$
$$+(1/2-3/2 \sin^2 \lambda) \sin^2 \omega(1/2-5/4\ e_1^2) \cos 2h\}. \qquad \mathrm{Ssa}$$
$$(69)$$

111. *Tide depending on fourth power of moon's parallax.*—This may be derived by substituting in the second term of equation (20) an

expression for $\cos\theta$ derived as explained in paragraph 107 and reducing, by the general method pursued in determining the tides due to the third power of the moon's parallax. All of the resulting terms are very small, the only one recognized being:

$$y=3/2\,(Ma^4/Ec^4)a[5/12\cos^3\lambda\cos^6 \tfrac{1}{2}I\cos(3T+3h-3s+3\xi-3\nu)].\ M_3 \quad (70)$$

112. *Equilibrium argument.*—Each term of equation (68) contains the general factor $3/2\ (Ma^3/Ec^3)a$, which has a constant value of 1.7527 feet; a factor composed of a function of the latitude of the tidal station, which is constant at a given station; a factor composed of a function of I, which changes very slowly, a constant factor containing e and m, and the cosine of an angle formed by the algebraic sum of simple multiples of the angles T, h, s, p, ξ, *and* ν. This angle is called the *equilibrium argument.* The term in equation (70) is in the same form but with a different general factor.

Similarly each term of equation (69) contains the general factor $3/2\ (Sa^3/Ec_1^3)a$, which has the constant value of 0.8091 feet; a factor composed of a function of the latitude of the tidal station; a factor composed of a function of ω, which does not change; a constant factor containing e_1 and the cosine of an equilibrium argument containing T, h, and p_1 only.

113. Since T is the hour angle of the mean sun at the tidal station, it is zero at noon, mean local time, at the station, and increases at the rate of 15° per mean solar hour.

The values of the angles h, s, p, and p_1, the longitudes of the mean sun, moon, and lunar and solar perigee, respectively, at the beginning of each calendar year at Greenwich are given in Manuals on Harmonic Analysis of Tides. Their rates of change remain practically constant for a century of time, and are as follows:

Angle	Angular change in degrees per mean solar hour	
	Symbol	Value
T	θ (theta)	15.
h	η (eta)	.041, 068, 64.
s	σ (sigma)	.549, 016, 53.
p	$\tilde{\omega}$ (omega)	.004, 641, 83.
p_1	$\tilde{\omega}_1$	.000, 001, 96.

That part of the equilibrium argument made up of the angles T, h, s, p, and p_1 which change at a constant rate, together with any constant term formed by the introduction of 90° or 180°, is conventionally represented by the symbol V. This part then has the form:

$$V=n_1T+n_2h+n_3s+n_4p+n_5p_1+n_690° \quad (71)$$

where n_1, n_2, n_3, etc., are small positive or negative integers, or zero.

114. The remaining part of the equilibrium argument is made up of simple multiples of the angles ν and ξ, the longitude and right ascension of the intersection, represented by the arcs UI and U_1I, figure 28. The arc UO is the longitude of the moon's node, conventionally represented as N; the angle $MOS=IOU$ is the constant angle i between the moon's orbit and the ecliptic, and the angle IUO is the constant angle ω between the equator and the ecliptic. The value of ν in terms of N and these known angles may therefore be determined by the solution of the spherical triangle IOU and the value of ξ then found from the right spherical triangle IUU_1. As the moon's node, O, makes the circuit of the ecliptic in its period of 19 years, the ascending intersection, I, moves to and fro over a comparatively small arc on either side of the vernal equinox, U, the angle ν increasing slowly from 0 to 13°.02, then decreasing to −13°.02 and increasing again to zero. The angle ξ similarly fluctuates between the limits of 11°.98 and −11°.98. The maximum change in these angles during a year is about 5°. The slowly fluctuating part of the equilibrium argument formed by these two angles is conventionally designated by the symbol u. The total equilibrium argument is then represented by $V+u$.

The value of N at the beginning of each calendar year at Greenwich is tabulated with those of h, s, p, and p_1. Its rate of change is −19°.326,19 per calendar year, or −0°.002,206,41 per mean solar hour. Its value at any instant is therefore readily found. The values of ν and ξ at that instant can then be found from a table giving these angles for each degree of N.

115. *Components of the equilibrium tide.*—If V_0 is the value of V at any given instant, taken as the origin of time, then at any time t thereafter,

$$V=V_0+at,$$

in which a is a constant whose value is:

$$a=n_1\theta+n_2\eta+n_3\sigma+n_4\tilde{\omega}+n_5\tilde{\omega}_1 \tag{72}$$

Each term of equations (68), (69), and (70) then has the form:

$$y=A\cos(V+u)=A\cos[at+(V_0+u)] \tag{73}$$

The form of this expression shows at once that each term represents a component of the equilibrium tide. For lunar components the values of A and u change slowly with the longitudes of the moon's node, but may be considered as constant during a limited period of time such as a month or even a year. For solar components, A is constant and u is zero.

116. The numerical value of the speed of each component of the equilibrium tide may be readily computed from the speeds of the

constituents of V given in paragraph 113. Thus the speed of the component represented by the first term of equation (68), viz,

$$3/2(Ma^3/Ec^3)a \cos^2 \lambda \cos^4 \tfrac{1}{2}I(1/2-5/4\ e^2) \cos (2T+2h-2s+2\xi+2\nu)$$

is

$$a=2\theta+2\eta-2\sigma=30°+0°.082,137,28-1°.098,035,06=28°.984,104,22$$

This is then the M_2 component of the equilibrium tide, its speed being identical with that previously identified for that component (par. 75). All of the other terms in equations (68), (69), and (70) may be similarly identified as the equilibrium components corresponding to the components listed in paragraphs 75 and 76. Their conventional symbols, conforming to those previously listed, are shown opposite each term. It will be noted that the semidiurnal components are those whose arguments contain the term $2T$; the diurnal components are those whose arguments contain the term T, and the long-period components are those in whose arguments T does not enter. The lunar and solar components designated as K_2 and K_1, each have the speed of $2\theta+2\eta$ and $\theta+\eta$, respectively. As previously pointed out, these pairs each unite into a single component. Their symbols are therefore enclosed in brackets to indicate that they are parts of a combined component. Two other lunar components, L_2 and M_2, appear twice in the list in brackets. The speed of the L_2 component represented by the third term in equation (68) is $2\theta+2\eta-\sigma-\tilde{\omega}=$ 29°.528,478,92 and that of the ninth term is $2\theta+2\eta-\sigma+\tilde{\omega}=$ 29°.537,762,58. The difference in these speeds is evidently $2\tilde{\omega}=0.00928366$ and the synodic period of the two components (par. 90) is 15/0.009,283,66 days=1,720 days. They therefore cannot be separated by analysing observations over a period of even a year, and consequently are treated as a single component. The evaluation of the coefficients of the two terms shows that the first is the larger, and its speed is therefore assigned to both. The speed of the M_1 component represented by the twelfth term is similarly $\theta+\eta-\sigma-\tilde{\omega}$ while that of the eighteenth term is $\theta+\eta-\sigma+\tilde{\omega}$. Since the difference in these speeds is also $2\tilde{\omega}$ they also cannot be separated by a year's observations. For convenience they are treated as a single component having a speed of 14°.492,052,1 whose component hour is the same as that of the principal lunar diurnal component M_1. The speed of the lunar fortnightly component MSf is exactly the same as that of a compound tide whose speed is the difference of the speeds of the M_2 and S_2 components, and this component is therefore also bracketed.

117. *Determination of the epoch of a component of the actual tide.*—As shown in paragraph 103, the phase of a component of the actual

tide differs from that of the corresponding equilibrium component by a fixed angle, which is designated as its epoch, and is conventionally represented by the symbol κ. If then the equation of the equilibrium component is (equation (73)):

$$y=A_1 \cos [at+(V_0+u)]$$

the equation of the component of the actual tide is:

$$y=A \cos [at+(V_0+u-\kappa)] \tag{74}$$

Comparing this equation with the equation for a component of the actual tide in the form given in equation (29):

$$y=A \cos (at-\zeta), \tag{75}$$

it is evident that:

$$V_0+u-\kappa=-\zeta, \tag{76}$$

whence:

$$\kappa=V_0+u+\zeta \tag{77}$$

The computation of ζ from a series of tidal observations was shown in paragraphs 94 and 95, the origin of time being taken at the beginning of the series. To determine the value of κ, the value of V_0 at the same origin of time must be computed. Since u is regarded as constant during the period of the observations, its value is taken as that at the middle of the period.

118. *Computation of* V_0+u.—For simplicity, the hourly tidal heights from which the components of the actual tide are computed begin at 0 hour (midnight). The time is usually the standard time at a time meridian, whose longitude, S, differs from the longitude, L, of the tidal station. Taking longitude west of Greenwich as positive, and east as negative, the Greenwich time of the beginning of the observations is then the $S/15$th hour of the initial date. The expression for V_0+u for each component is in the form:

$$V_0+u=n_1T_0+n_2h_0+n_3s_0+n_4p_0+n_5(p_1)_0+n_6 90^\circ+n_7\xi_1+n_8\nu_1 \tag{78}$$

in which T_0, h_0, s_0, p_0, and $(p_1)_0$ are the values of the respective angles at 0 hour on the initial date of the observations, and ξ_1, and ν_1 are the values of ξ and ν at the middle of the period.

Since T, the hour angle of the mean sun, is zero at noon, mean local time of the tidal station, it is $\pm 180^\circ$ at midnight (0 hour) mean local time, and $(S-L)\pm 180^\circ$ at midnight, standard time. For dirunal components, $n_1=1$, and $n_1T_0=S-L\pm 180^\circ$; for semidiurnal components $n_1=2$, and $n_1T_0=2(S-L)\pm 360^\circ=2(S-L)$.

As has been stated, the values of h, s, p, p_1, and N, at 0 hour Greenwich mean civil time on January 1 of each calendar year are given in tables contained in manuals for the harmonic analysis of tides, together with the differences to be added successively to give these values on the first day of each month, on each day of the month, and at each hour of the day, Greenwich time. The values of h_0, s_0, p_0, and $(p_1)_0$ are the values of h, s, p, and p_1, at Greenwich time of 0 hour on the initial date of the observations. The value of N, the longitude of the moon's node, is similarly taken off for the middle of the period of observations, and from it the values of ξ, and ν, taken from the table showing the value of these angles for each degree of N. Entering these values in equation (78), the value of V_0+u is immediately determined. This value, added (algebraically) to the value of ζ found from the observations, gives the value of κ. The value of κ so derived is, it may be observed, independent of the meridian on which the times of the observations are based.

If the observations are made on local time, instead of standard time, $S=L$, and the angle $S-L$ becomes zero.

119. *Example.*—The computation in paragraph 95 shows that for the M_2 component of Sitka, Alaska, long. 135°20′ W., for the 29-day period beginning at 0 hour, mean local time, July 1, 1893, $\zeta=61°.6$. The Greenwich time of the beginning of the period is then $9.02h$, July 1. The equilibrium argument for the M_2 component is from equation (68):

$$V+u=2T+2h-2s+2\xi-2\nu.$$

Since the observations are on mean local time, $2T=0$. The values of h and s, at $9.02h$, July 1, 1893, Greenwich time, found from the tables, are:

$$h_0=99°.64 \qquad s_0=308°.03$$

and the value of N for $9.02h$ July 15, 1893, is 24°.17. The corresponding tabular values of ν_1 and ξ_1 are:

$$\nu_1=4°.45 \qquad \xi_1=4°.01.$$

Then

$$V_0+u=0+199°.28-256°.06+8°.02-8°.90=-57°.66=302°.34$$

$$\kappa=61°.6+302°.3-360°=3°.9.$$

The value of κ derived from observations for a year is 3°. (Table V, par. 134.)

120. *Greenwich epochs.*—The Greenwich epoch of a component at a station is the difference between the phases of the equilibrium component at Greenwich and of the actual component at the station.

At any given origin of Greenwich time, the initial phase of the equilibrium component at Greenwich is, since S and L are both zero:

$$V_0+u=n_1(\pm 180°)+n_2h_0+n_3s_0+n_4p_0+n_5(p_{\prime})_0+n_6 90°+u \qquad (79)$$

while the initial phase of the actual tide at a station whose latitude is L is:

$$V_0+u-\kappa=n_1(-L\pm 180°)+n_2h_0+n_3s_0+n_4p_0+n_5(p_{\prime})_0+n_6 90°+u \qquad (80)$$

The difference is:

$$G=n_1L+\kappa. \qquad (81)$$

It may be observed that n_1 is the same as the subscript of the component. The formula for Greenwich epoch is usually written:

$$G=pL+\kappa. \qquad (82)$$

In which G is the Greenwich epoch, p the subscript of the component, L the west longitude of the station and κ the local epoch.

The difference between the Greenwich epochs of a component of the tide at any two stations is the constant difference between the phases of the component at the two stations.

121. *Equilibrium arguments of overtides and compound tides.*—The equilibrium argument of an overtide is taken as the indicated multiple of that of the primary tide. The equilibrium arguments of compound tides are the sums or differences of those of the components compounded.

122. *Expression for u of the* K_1, K_2, L_2, *and* M_1 *components.*—A reference to equations (68) and (69) shows that the K_1 component is the resultant of a lunar component whose equilibrium argument is $T+h-\nu-90°$ and a solar component whose argument is $T+h-90°$.

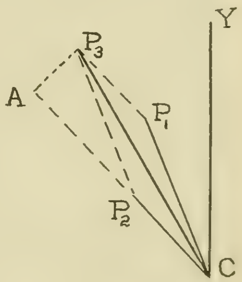

FIGURE 29.

The relation of the resultant to the components is graphically shown in figure 29, in which CP_1 is the amplitude of the lunar component, CP_2 the amplitude of the solar component, and CP_3 the amplitude of the resultant. The angle YCP_1 is $T+h-90°-\nu$, the angle YCP_2 is $T+h-90°$, and the angle P_1CP_2 is ν. Placing the angle $P_3CP_2=\nu'$ the equilibrium argument of the resultant is $T+h-90°-\nu'$. If A is the foot of the perpendicular drawn from P_3 to CP_2 produced, then

$$\sin \nu'=\overline{AP}_3/\overline{CP}_3=\overline{P_2P}_3 \sin \nu\,/\overline{CP}_3=\overline{CP}_1 \sin \nu\,/\overline{CP}_3 \qquad (83)$$

$$\cos \nu'=(\overline{CP}_2+\overline{P_2A})/CP_3=(\overline{CP}_2+\overline{CP}_1 \cos \nu)/\overline{CP}_3. \qquad (84)$$

whence

$$\tan \nu' = \overline{CP}_1 \sin \nu / (\overline{CP}_2 + \overline{CP}_1 \cos \nu) = \sin \nu / (\cos \nu + \overline{CP}_2 / \overline{CP}_1) \qquad (85)$$

The amplitude CP_2 is, from equation (69), after substituting the numerical value of the general factor of this equation (par. 112):

$$0.8091(1/4+3/8\ e^2_1) \sin 2\omega \sin 2\lambda,$$

and the amplitude CP_1 is, from equation (68):

$$1.7527(1/4+3/8\ e^2) \sin 2I \sin 2\lambda$$

The substitution of these values in equation (85) gives, after applying the numerical values of e, e_1 and ω

$$\nu' = \tan^{-1} \frac{\sin \nu \sin 2I}{\cos \nu \sin 2I + 0.3357} \qquad (86)$$

The equilibrium argument for the K_2 component may similarly be shown to be: $V+u=2T+2h-2\nu''$, where

$$2\nu'' = \tan^{-1} \frac{\sin 2\nu \sin^2 I}{\cos 2\nu \sin^2 I + 0.0728} \qquad (87)$$

Since ν and I are both functions N, the longitude of the moon's node, ν' and $2\nu''$ are also functions of N. The values of ν' and $2\nu''$ for each degree of N are included in the tables showing the values of ν and ξ (par. 114).

The equilibrium arguments for the L_2 and M_1 components are taken from special tables, contained in Manuals for the Harmonic Analysis of Tides. These components are not important, and the derivation and application of these tables is here omitted.

MEAN VALUES

123. *Equilibrium components.*—Each component of the lunar equilibrium tide developed in equation (68) is in the form:

$$y = J \cos (V+u) \qquad (88)$$

in which J is made up of factors formed by astronomical constants; a factor formed of a trigonometric function of λ, the latitude of the tidal station, which is constant at any given station; and a factor formed of a trigonometric function of I, the inclination of the moon's orbit to the Equator, which slowly changes with the longitude of the moon's node. If this last factor is represented by ϕ (I), then

$$J = C\ \phi(I) \qquad (89)$$

In the first term of equation (78), for example:

$$C=3/2(Ma^3/Ec^3)a(1/2-5/4\ e^2)\cos^2\lambda, \text{ and } \phi(I)=\cos^4\tfrac{1}{2}I.$$

The amplitude, J, of the equilibrium component fluctuates slowly between fixed limits with the changing values of I. A rigid analysis, which need not be here repeated, shows that the mean value of J during the circuit of the moon's node around the ecliptic, is

$$J_0=C\,[\phi(I)]_0\times[\cos u]_0=CM \tag{90}$$

where $[\phi(I)]_0$ is the mean value of $\phi\ (I)$, $[\cos u]_0$ is the mean value of $\cos u$, and M is the numerical value of their product.

From equations (89) and (90)

$$J_0/J=M/\phi(I) \tag{91}$$

124. *Mean value of amplitudes of the actual components.*—As a basic assumption, the fluctuation of the amplitude of a component of the actual tide with the changing values of I is proportional to the concurrent fluctuation of the corresponding equilibrium component (par. 102). If then R is the value of the amplitude of a component of the actual tide as determined from a particular set of observations, H the mean value of the amplitude, and J the amplitude of the corresponding component of the equilibrium tide when I has the value prevailing during the period of the observations:

$$H/R=J_0/J=M/\phi(I) \tag{92}$$

The factor $M/\phi(I)$ is conventionally designated as F. Its reciprocal, $\phi(I)/M$ is designated as f. Hence

$$H=FR \qquad R=fH. \tag{93}$$

125. *Expressions for F.*—A reference to equation (68) shows that the expressions for $\phi\ (I)$ and for u in the terms representing the various components are as follows:

Components	$\phi(I)$	u
M_2, N_2, 2N, ν_2, λ_2, μ_2,	$\cos^4\frac{1}{2}I$,	$2\xi-2\nu$
O_1, Q_1, 2Q, ρ_1,	$\sin I\cos^2\frac{1}{2}I$,	$2\xi-\nu$
OO,	$\sin I\sin^2\frac{1}{2}I$,	$-(2\xi+\nu)$
J_1,	$\sin 2\ I$,	$-\nu$
Mf,	$\sin^2 I$,	-2ξ
Mm,	$1-3/2\sin^2 I$,	0

The mean values of these functions of I, and of the corresponding expressions for $\cos u$ are found by deriving the expressions for I, ξ, and ν in terms of N from the spherical triangles OUI and IUU_1, figure

28; transforming the functions of I, and the expressions of u above listed into functions of N; and finding the mean value of these expressions as N varies from 0 to 360°. This somewhat lengthy derivation, which need not be here repeated, shows that the products of these mean values are as follows:

Function	$[\phi(I)]_0[\cos u]_0$	M, numerical value
$\cos^4 \frac{1}{2}I$	$\cos^4 \frac{1}{2}\omega \cos^4 \frac{1}{2}i$	0.9154
$\sin I \cos^2 \frac{1}{2}I$	$\sin \omega \cos^2 \frac{1}{2}\omega \cos^4 \frac{1}{2}i$	.3800
$\sin I \sin^2 \frac{1}{2}I$	$\sin \omega \sin^2 \frac{1}{2}\omega \cos^4 \frac{1}{2}i$	.0164
$\sin 2I$	$\sin 2\omega\ (1-3/2 \sin^2 i)$	.7214
$\sin^2 I$	$\sin^2 \omega \cos^4 \frac{1}{2}i$	.1578
$1-3/2 \sin^2 I$	$(1-3/2 \sin^2 \omega)\ (1-3/2 \sin^2 i)$	.7532

The numerical values in the last column are found by substituting the values of $\omega=23°.452$; and $i=5°.145$ in the expressions in the second column. It may be noted that as the moon's orbit tilts to and fro, the median value of its inclination I to the Equator is the inclination ω of the ecliptic to the Equator. Since the values of $\cos^4 \frac{1}{2}i$ and of $(1-3/2 \sin^2 i)$ are very close to unity, the mean values differ but little from the value of the function when $I=\omega$.

The expressions for the reduction factor F are then:

For M_2, N_2, 2N, ν_2, λ_2, and μ_2,	$F=0.9154/\cos^4 \frac{1}{2}I$	
For O_1, Q_1, 2Q, and ρ_1,	$F=0.3800/\sin I \cos^2 \frac{1}{2}I$	
For OO,	$F=0.0164/\sin I \sin^2 \frac{1}{2}I$	
For J_1,	$F=0.7214/\sin 2I$	
For Mf,	$F=0.1578/\sin^2 I$	
For Mm,	$F=0.7532/(1-3/2 \sin^2 I)$	(94)

126. The reduction factors for the lunisolar components K_1 and K_2 are more lengthy functions of I, and those for the L_2 and M_1 components are still more complicated. The derivation of these factors is explained in full in Special Publication No. 98; United States Coast and Geodetic Survey, and is not here described.

127. *Application of reduction factors.*—The logarithms of the reduction factors for the several lunar and lunisolar components, corresponding to each tenth of a degree of I, are tabulated in Manuals on the Harmonic Analysis of Tides, special tables being included from which the factors for the L_2 and M_1 may be found. To find the mean value of the amplitude of a component from the value determined from a particular set of observations, the value of N at the middle of the period is taken off as described in paragraph 118, the corresponding value of I taken from a table, and from it the logarithm of F ascertained. Thus the amplitude of the M_2 component at Sitka, Alaska, for the 29-day period beginning July 1, 1893, corrected by the augmenting factor, was found in paragraph 99 to be, $R=3.430$ feet.

The value of N at the middle of the period was found in paragraph 119 to be 24°.17. The corresponding tabular value of I is 28°.22; and for this value the tabular value of log F for the M_2 component is 0.0148.

Then

$$\begin{aligned} \log R &= 0.5353 \\ \log F &= .0148 \\ \hline \log H &= .5501 \\ H &= 3.549 \end{aligned}$$

The value of H derived from a year's observation is 3.591 (table V, par. 134).

Since solar components do not vary with I, no reduction factor is to be applied to them.

128. *Reduction factors for other components.*—It has been seen that for the M_2 component

$$\phi(I)=\cos^4 \tfrac{1}{2}I \qquad u=2\xi-2\nu$$

The corresponding expressions for the M_3 component are, from equation (80):

$$\phi(I)=\cos^6 \tfrac{1}{2}I \qquad u=3\xi-3\nu$$

Since $\cos^6 \tfrac{1}{2}I=(\cos^4 \tfrac{1}{2}I)^{3/2}$, it may be presumed that the reduction factor for the M_3 component is

$$F=(F \text{ of } M_2)^{3/2}$$

and this relation is established by a detailed analysis.

Similarly the reduction factor for the lunar overtides are taken as the squares, cubes, etc., of the fundamental tide. These factors are then:

For M_4, $F=(F \text{ of } M_2)^2$; M_6, $F=(F \text{ of } M_2)^3$, and so on.

No reduction factors are to be applied to the solar overtides.

The factors for compound tides are taken as the products of the factors of the components compounded, the factor for any solar component entering into the compound tide being unity.

129. *"Mean values of coefficients."*—An examination of equations (68) and (69) shows that the amplitude of each semidiurnal component of the equilibrium tide is the product of a coefficient, whose numerical value may be determined from astronomical data, times $\cos^2\lambda$; the amplitude of each diurnal component is a coefficient times $\sin 2\lambda$; and the amplitude of each long period component a coefficient times $(1/2—3/2 \sin^2\lambda)$. The mean values of these coefficients therefore show the relative magnitudes of the mean values of the amplitudes of the semidiurnal, diurnal, and long-period equilibrium components, respectively, at a given station. The "mean values of the

coefficients'' conventionally used, and shown in table IV at the end of this paragraph, are the complete coefficients divided by 1.7527, the numerical value of the general factor 3/2 $(Ma^3/Ec^3)a$, in equation (68); but these afford an equally good measure of the relative magnitudes of the mean amplitudes of the equilibrium components in the three classes. Thus the mean value of the coefficient of the M_2 component is $(1/2—5/4\,e^2)M$, in which M has the numerical value of 0.9154 derived in paragraph 125. The mean value of the coefficient of the S_2 component is $G(1/2—5/4\,e^2{}_1)\cos^4 \tfrac{1}{2}\omega$, in which G is the ratio of the general factor in equation (69) to the general factor in equation (68), this ratio being 0.46164.

TABLE IV.—*Mean value of coefficients*

Semidiurnal		Diurnal		Long period	
M_2	0.4543	K_1	0.2655		
S_2	.2120	O_1	.1886	Mf	0.0763
N_2	.0880	P_1	.0880	Mm	.0414
K_2	.0576	Q_1	.0365	Msf	.0042
L_2	.0126	M_1	.0149	Ssa	.0365
T_2	.0124	J_1	.0149		
ν_2	.0123	OO	.0081		
2N	.0117	ρ_1	.0051		
μ_2	.0074	2Q	.0049		
R_2	.0018				
λ_2	.0018				

INFERENCE OF AMPLITUDES AND EPOCHS

130. It has been found that the mean values of the amplitudes of the semidiurnal components of the actual tides at any station are generally proportional to the mean values of the amplitudes of their corresponding equilibrium components, as are the amplitudes of the diurnal components. The amplitudes of the minor components may therefore be approximately determined from those of the larger components of the same type by applying this proportion, without going through the laborious process of determining them by harmonic analysis. The ratio of the amplitudes of the equilibrium components is given by the ratio of the ''mean values of the coefficients'' listed in table IV. It has also been found that the difference in the epochs of components of the same type is proportional to the difference in their speeds. Thus if κ_1, κ_2, κ_3 are the epochs of three components, and a', a'', and a'' their speeds

$$(\kappa_3-\kappa_1)/(\kappa_2-\kappa_1)=(a'''-a')/(a''-a')$$

whence

$$\kappa_3=\kappa_1+(\kappa_2-\kappa_1)\ (a'''-a')/(a''-a') \tag{95}$$

If then the epochs κ_1 and κ_2 have been determined by analysis, the epoch κ_3 can be determined by inference.

131. In the harmonic analysis of small components, accidental variations in the tidal heights may conceal, to a relatively large meas-

ure, the systematic variation sought for, and the amplitudes and epochs derived by inference may be preferable to those determined by direct analysis, particularly if the period of observations is short. A considerable number of these small components are customarily determined by inference.

SUMMARY OF THE METHOD OF HARMONIC ANALYSIS

132. The harmonic analysis of the tide at a station comprises:

(*a*) Some six or more separate summations of the observed hourly tidal heights for a period generally of 369 days to obtain the hourly component heights of the S, M, and K group of components, and the larger individual components (pars. 78 to 89).

(*b*) The computation of V_0 for each component at the initial hour of the observations and of u at the middle of the period (par. 118).

(*c*) The preliminary determination of the epochs, κ, and of the amplitudes, R (corrected by the augmenting factor) of the components of each group, and the larger individual components, from their computed hourly component tidal heights (pars. 91–99, and 118), and the preliminary inference of the remainder for use in elimination (par. 130).

(*d*) The elimination of the effect of one component on another (par. 100).

(*e*) The reduction of the corrected amplitudes to their mean values, H (par. 127) and the final inference of the constants of the components not analyzed.

133. Standard forms to systematize these computations, and tables giving the requisite data are published in the Manual of Harmonic Analysis and Prediction of Tides of the United States Coast and Geodetic Survey. The labor entailed in the analysis of the tide at a station is apparent. The dependability of the results of a tidal analysis is illustrated by a comparison between the separate determinations of the harmonic constants at Fort Hamilton, New York Harbor, for three periods of 369 days beginning January 1, 1900, 1904, and 1928, respectively. Omitting the constants derived by inference, the determinations are as follows:

Component	Amplitude H, in feet			Epoch κ, in degrees		
	1900	1904	1928	1900	1904	1928
M_2	2.212	2.208	2.256	221°.3	220°.7	220°.7
N_2	.459	.496	.473	203°.9	204°.5	202°.1
S_2	.440	.450	.461	248°.8	247°.0	248°.6
K_1	.320	.324	.316	103°.1	104°.0	102°.5
O_1	.178	.167	.171	98°.1	98°.9	100°.5
P_1	.102	.095	.096	102°.7	109°.1	108°.8
K_2	.148	.132	.120	244°.1	235°.8	252°.1
M_1	.008	.007	.012	86°.8	123°.0	123°.0
M^4	.028	.030	.055	333°.4	345°.3	313°.6
M_6	.051	.053	.063	35°.8	34°.9	36°.8
S_1	.044	.036	.050	68°.5	58°.6	60°.1
S_4	.035	.042	.030	75°.8	63°.9	59°.8

134. *Principal tidal components at representative stations.*—The harmonic components of the tide have been determined at a large number of tidal stations throughout the world. The amplitudes, in feet, and the epochs, in degrees, of the five principal components at stations used in the ensuing chapters to illustrate the characteristics of the tide and the determination of tidal datums, are abstracted as follows from the extensive data given in Special Publication, No. 98, United States Coast and Geodetic Survey (1924).

TABLE V.—*Principal tidal components*

Station	M_2	$M_2°$	S_2	$S_2°$	N_2	$N_2°$	K_1	$K_1°$	O_1	$O_1°$
Eastport, Maine	8.576	326	1.399	6	1.725	298	0.480	129	0.377	111
Pulpit Harbor, Maine	4.899	320	.777	355	1.049	288	.457	129	.365	108
Portland, Maine	4.372	324	.699	0	.949	292	.462	132	.353	111
Boston, Mass	4.371	330	.699	5	.995	300	.449	134	.348	117
Fort Hamilton, New York Harbor	2.210	221	.445	248	.478	204	.322	104	.172	98
Fernandina, Fla	2.854	228	.509	258	.585	213	.345	127	.252	129
Galveston, Tex	.308	108	.094	111	.074	91	.386	315	.358	309
Cristobal, C. Z	.268	6	.044	197	.087	329	.337	161	.196	160
Balboa, C. Z	6.000	89	1.616	146	1.260	58	.443	343	.128	355
Presidio, San Francisco, Calif	1.773	330	.404	335	.376	301	1.208	106	.756	89
Seattle, Wash	3.494	128	.846	154	.686	97	2.697	156	1.502	133
Ketchikan, Alaska	6.138	7	2.014	38	1.241	342	1.648	129	1.014	114
Sitka, Alaska	3.591	3	1.145	34	.758	335	1.504	125	.905	110
Sheerness, England	6.297	1	1.750	56	1.046	337	.377	14	.451	193
Do Son, Indochina	.131	113	.098	140	.025	99	2.362	91	2.297	35

HARMONIC PREDICTION OF THE TIDES

135. The chief use made of the harmonic constants is in the prediction of the heights and times of high and low waters at a tidal station.

The height of the tide at any time during a particular year is given by the equation:

$$y = H_0 + f\mathrm{J}_1 \cos(\mathrm{j}_1 t + V_0 + u - \mathrm{J}_1°) + f\mathrm{K}_1 \cos(\mathrm{k}_1 t + V_0 + u - \mathrm{K}_1°) + \cdots$$
$$+ f\mathrm{K}_2 \cos(\mathrm{k}_2 t + V_0 + u - \mathrm{K}_2°) + \cdots \quad (96)$$

in which H_0 is the height of the mean sea level above any standard reference plane,

J_1, K_1, K_2, etc. are the mean values of the amplitudes of the several tidal components,

f is the mean value during the year (taken as the value at the middle of the year) of the factor to reduce the mean value of the amplitude to its value for the year (par. 124),

t is the time in hours after the beginning of the year,

j_1, k_1, k_2, etc., are the speeds of the several components,

V_0 is the value of V for the component at the beginning of the year,

u is the value of u for the component at the middle of the year,

$\mathrm{J}_1°$, $\mathrm{K}_1°$, $\mathrm{K}_2°$, etc., are the epochs of the components.

The times of high water and of low water are those at which y is a maximum or minimum respectively, and therefore at which $dy/dt=0$. They are then given by the equation:

$$dy/dt = -\mathrm{j}_1 f\mathrm{J}_1 \sin(\mathrm{j}_1 t + V_0 + u - \mathrm{J}_1°) - \mathrm{k}_1 f\mathrm{K}_1 \sin(\mathrm{k}_1 t + V_0 + u - \mathrm{K}_1°)$$
$$- \cdot \cdot \cdot - \mathrm{k}_2 f\mathrm{K}_2 \sin(\mathrm{k}_2 t + V_0 + u - \mathrm{K}_2°) + \cdot \cdot \cdot = 0 \qquad (97)$$

136. *Tide-predicting machine.*—The arithmetic evaluation of y for successive values of t from equation (96) would be too laborious to be practicable. The solution of equation (97) for the successive values of t at high and low water could be accomplished mathematically only by an even more laborious process of successive approximations. An elaborate machine, called the tide-predicting machine, has been devised and constructed, by which the values of y and of dy/dt in these equations are mechanically summed for values of t measured by the angular travel of the mechanism. The height of mean sea level, H_0, the values, fH, of the amplitudes of the several components for the year of prediction, and the values of the initial phases, $V_0+u-\kappa$ of the several components at the beginning of the year, are all set on the dials of the machine. The machine is then put in motion. When the pointer indicating the value of dy/dt (the sine summation) crosses its zero mark, the machine is stopped, the height of the tide is read off the dial which indicates the summation of y, and the time of the tide is read off a dial which indicates the time corresponding to the angular travel of the mechanism. These are the height and time of the first high or low water of the year. The machine is again set in motion, the height and time of the next low or high water read off, and the process continued until the predictions for the year are completed.

137. Tide Tables published annually by the Coast and Geodetic Survey give the predicted heights and times of the tides at some one hundred reference stations throughout the world, with data showing the corrections to be applied to give these heights and times at numerous secondary stations.

138. *Accuracy of tide predictions.*—The predicted times of high and low water which are published in the tide tables, obviously must be those which would occur without the accidental disturbances due to winds and other meteorological causes. A comparison between the actual and the predicted tides at Portland, Maine, and at Seattle, Wash., in May and November 1919, shows that the maximum departure in the time of the actual from the predicted high and low waters was 24 minutes; but that the times were generally in much closer agreement. The height of one of the tides at Portland differed by 1.9 feet from the predicted height; and it was not unusual for the

tides at both stations to differ by more than half a foot from those predicted; but most of the observed heights were within half a foot of the predictions (The Tide, Marmer, p. 205, et seq.).

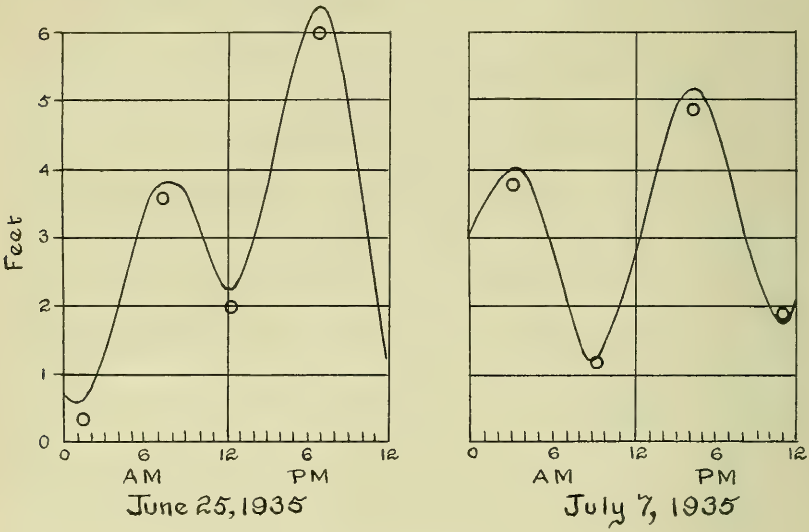

FIGURE 30.—Recorded and predicted tides, San Francisco (predicted tides enclosed in circles).

The plot in figure 30 of the recorded tides and the predicted high and low waters at San Francisco, Calif., on 2 days chosen at random, indicates the correspondence ordinarily to be expected.

Chapter III

CHARACTERISTICS OF THE TIDES

139. *Types of tides.*—The tides throughout the world are of three general types, which are determined by the relative magnitude of the semidiurnal part of the tide (as indicated by the amplitudes of the principal semidiurnal components, M_2, S_2, and N_2) and of the diurnal part of the tide (as indicated by the principal diurnal components, K_1 and O_1). These types are:

(*a*) *The semidiurnal or semidaily tides.*—Tides of this type have two nearly equal high waters and two nearly equal low waters each lunar day of 24 hours 50 minutes. They occur when the amplitudes of the diurnal components are small in comparison with those of the semidiurnal components. This type is found along both coasts of North and South Atlantic Oceans, and at other places as well.

(*b*) *Mixed tides.*—This type is characterized by two markedly unequal high waters, or two markedly unequal low waters, or both, on each lunar day, during most of the month. Tides are of this type when the amplitudes of the diurnal components are considerable in comparison with those of the semidiurnal components, but do not greatly exceed the latter. Such tides are common, but not universal, along the coasts of the Pacific Ocean.

(*c*) *Diurnal tides.*—Tides of this type have but one high water and but one low water each day during a substantial part of or all of the month. Such tides are common along the coasts of large enclosed seas with restricted entrances, such as the Gulf of Mexico, the Caribbean, the waters of the East Indies, and the Mediterranean; and sometimes at oceanic islands. Diurnal tides are usually quite small and irregular.

140. Tides of the semidiurnal type usually have some diurnal inequality during the two periods in each month when the moon is

farthest from the Equator and the diurnal tidal impulses are consequently a maximum (paragraph 40). In tides of the mixed type, the two daily high waters and the two daily low waters become nearly equal when the moon is near the Equator, and the diurnal tidal impulses are a minimum. When the moon is at its maximum declination, near the celestial tropics, one of the two daily low waters or high waters of tides of the mixed type occasionally may disappear, producing a diurnal tide. On the other hand, tides of the diurnal type usually break down into two daily tides during a part of the month, although in the Gulf of Tongking in Indo-china, the tide remains diurnal throughout the month.

Obviously the types of tide merge into each other. The accepted criterion distinguishing the types is the ratio $(K_1+O_1)/(M_2+S_2)$, derived from the harmonic components at the station. If this ratio is less than 0.25, the tide is classed a semidirunal; if between 0.25 and 1.25 as mixed; and if over 1.25 the tide is classed as diurnal.

THE EFFECT OF THE PRINCIPAL SEMIDIURNAL COMPONENTS ON THE TIDES

141. *The* M_2 *component, semidiurnal tides.*—When the tide is of the semidiurnal type, the M_2 component, with rare exceptions, is the dominant one, with an amplitude nearly but not quite one-half of the mean tidal range. Generally, its amplitude may be taken as 0.47 times the mean range.

142. *Relation of epoch of* M_2 *component to lunitidal intervals.*—As has been seen (paragraph 117), the expression for the M_2 component of the actual tide may be written in the form:

$$y = M_2 \cos (m_2 t + V_0 + u - M_2{}^\circ) \tag{98}$$

in which

m_2 is the speed of the component, and has the numerical value of 28.984° per solar hour.

V_0+u is the value of the equilibrium argument at any arbitrarily chosen origin of time.

$M_2{}^\circ$ is the epoch of the component.

The expression for the M_2 equilibrium component is

$$y = M_2 \cos (m_2 t + V_0 + u) \tag{99}$$

At the high water of the actual tide

$$m_2 t + V_0 + u - M_2{}^\circ = 0$$

whence

$$t = [M_2{}^\circ - V_0 - u]/m_2 \tag{100}$$

Similarly, at each high water of the equilibrium tide

$$t = -(V_0 + u)/\mathrm{m}_2 \tag{101}$$

The nature of the equilibrium tide is such that the high waters of its M_2 component must occur at the moon's transits across the meridian of the tidal station. When the origin of time is taken at a lunar transit, equation (101) shows that V_0+u must be zero. The time of high water of the M_2 component of the actual tide is then, from equation (100), $M_2°/m_2$ hours after a lunar transit.

The M_2 component is the dominant one when the tide is of the semidiurnal type, and largely determines the time of high water of the entire tide. The other semidiurnal components, alternately advance and retard the time of high water. The diurnal components and lunar overtides may produce a systematic difference in the time of high water. Denoting this systematic difference by Δt, the *average* interval between a lunar transit and the time of high water at a station is then $(M_2°/m_2)+\Delta t$. This average interval is the *high-water interval* at the station (paragraph 8) and is designated as HWI. It follows therefore that:

$$\mathrm{M}_2° = \mathrm{m}_2(\mathrm{HWI} - \Delta t) \tag{102}$$

The low water of the M_2 component similarly occurs when:

$$\mathrm{m}_2 t - \mathrm{M}_2° = \pm 180°$$

or when

$$t = (\mathrm{M}_2° \pm 180°)/\mathrm{m}_2 \tag{103}$$

Since the diurnal components and lunar overtides retard (or advance) the time of low water by the same amount that they advance (or retard) the time of high water

$$\mathrm{M}_2° = \mathrm{m}_2(\mathrm{LWI} + \Delta t) \mp 180° \tag{104}$$

where LWI is the low-water interval at the station.

Combining equations (102) and (104) to eliminate Δt, and substituting for m_2 its numerical value:

$$\mathrm{M}_2° = 14°.492(\mathrm{HWI} + \mathrm{LWI}) \mp 90° \tag{105}$$

The negative sign is applied to the last term when the HWI is less than the LWI.

For example, at Fort Hamilton, New York Harbor, the high-water interval is 7.67 hours and the low-water interval is 1.64 hours. The epoch of the M_2 component, from formula (105), then is:

$$14°.492(7.67+1.64)+90° = 225°$$

Its values from harmonic analysis is 221°.

At Philadelphia, the high-water interval is 1.49 hours, and the low-water interval 8.97 hours. The epoch of the M_2 component from formula (105) then is

$$14°.492(1.49+8.97)-90°=62°$$

Its value from harmonic analysis is 49°.

Formula (105) gives evidently only an approximate value of the epoch of the M_2 component.

143. *The* S_2 *component—spring and neap tides.*—At stations having a tide of the semidiurnal type, the amplitude of the S_2 component is generally from one-sixth to one-half of that of the M_2 component. Since the difference in the speeds of these two components is relatively small, the resultant of the two fluctuates slowly from a maximum of M_2+S_2, when the generating radii of these components coincide, to a minimum of M_2-S_2 when they are 180° apart (par. 54), the period of the fluctuation, from maximum to minimum, being the synodic period of the two components, or $360°/(s_2-m_2)=354.367$ hours. This period is one-half of the lunar synodic month, the average interval from full moon to full moon.

Other tidal influences disregarded, the high and low waters occurring nearest the time at which the resultant of the M_2 and S_2 components is at a maximum are respectively higher and lower than at other times, and the tidal range is the greatest. These are the *spring* tides, and their range is the *spring range* (pars. 2 and 20). The tides nearest the time at which the resultant is a minimum are similarly the *neap* tides. The times at which the generating radii of the M_2 and S_2 components are in coincidence, and their resultant a maximum, may be called the time of spring tides, although this time is not generally the exact time of either spring high water or spring low water. Similarly, the time at which these generating radii are opposed may be called the time of neap tides. Because of the effect of the other components, the average spring range somewhat exceeds 2 (M_2+S_2) and the average neap range somewhat exceeds 2 (M_2-S_2).

144. *Phase age.*—The interval between the instant of full or new moon, and the time of spring tides is called the *phase age.* At the instant of full or new moon, the S_2 and M_2 components of the equilibrium tides quite evidently are in conjunction, and the difference in their phases is zero. Since the phases of the corresponding components of the actual tides differ from those of the equilibrium components by their respective epochs, $S_2°$ and $M_2°$, the difference in the phases of these components of the actual tides at the instant of full or new moon is $S_2°-M_2°$; and since the S_2 component gains on the M_2 component at the rate of $s_2-m_2°$ per hour, they are in conjunction

$(S_2^\circ - M_2^\circ)/(s_2 - m_2)$ hours after the time of full or new moon. Therefore:

Phase age (in hours)

$$= (S_2^\circ - M_2^\circ)/(30^\circ - 28^\circ.984) = 0.984 (S_2^\circ - M_2^\circ) \qquad (106)$$

It is easily shown that this expression gives also the time of neap tides after the instants at which the moon is in quadrature.

For example, at Fort Hamilton, New York Harbor, $S_2^\circ = 248^\circ$ and $M_2^\circ = 221^\circ$. The phase age is therefore 0.984 (248−221) hours = 26.5 hours. At this station therefore spring tides occur a little more than one day after the moon is at full or change (new), and neap tides at the same interval after the moon is at quadrature.

The phase age at tidal stations throughout the world ranges up to 3 days. It rarely is negative.

145. *The N_2 component—perigean and apogean tides.*—The amplitude of the N_2 component generally is between one-sixth and one-third of that of the M_2 component. At stations on the Atlantic coast of the United States, the N_2 component usually has a larger amplitude than the S_2 component, but at stations on the Atlantic coast of Europe, and along the British Isles, the amplitude of the N_2 component is materially less than that of S_2.

It is evident from the preceding paragraphs that the resultant of the M_2 and N_2 components fluctuates between a maximum of $M_2 + N_2$ and a minimum of $M_2 - N_2$, the period from maximum to maximum being

$$360^\circ (m_2 - n_2) = 360^\circ/(28.9841 - 28.4397) = 360^\circ/0.5444 = 661 \text{ hours.}$$

This is the length of the lunar anomalistic month (par. 62).

The maximum amplitude of the resultant obviously is due to the maximum attraction of the moon at perigee, and is called the *perigean tide.* Its minimum amplitude results from the minimum attraction of the moon at apogee, and is called the *apogean tide.* The average *perigean range* of the entire tide slightly exceeds 2 $(M_2 + N_2)$ and the average *apogean range* 2 $(M_2 - N_2)$.

146. *Parallax age.*—The interval between lunar perigee and the time of perigean tides is called the *parallax age.* Since the M_2 and N_2 components of the equilibrium tides are in conjunction at lunar perigee, the phases of these components of the actual tides then differ by the difference in their epochs, $M_2^\circ - N_2^\circ$; and these components of the actual tides come into conjunction $(M_2^\circ - N_2^\circ)/(m_2 - n_2)$ hours later. The expression for the parallax age is then:

$$\text{Parallax age (in hours)} = (M_2^\circ - N_2^\circ)/0.5444 = 1.837 \, (M_2^\circ - N_2^\circ) \qquad (107)$$

As is readily shown, the parallax age gives also the interval between lunar apogee and apogean tides.

For example, at Fort Hamilton, New York Harbor, $M_2° = 221°$ and $N_2° = 204°$. The parallax age is therefore

$$1.837\ (221° - 204°) = 31 \text{ hours.}$$

Perigean tides occur at this station, therefore, a little more than one day after lunar perigee; and apogean tides a little more than one day after lunar apogee.

The parallax age at stations throughout the world ranges up to 3 days. In some regions it has a negative value.

147. *Combined effect of* S_2 *and* N_2 *components.*—Perigean and apogean tides tend to obscure the spring and neap tides at stations at which

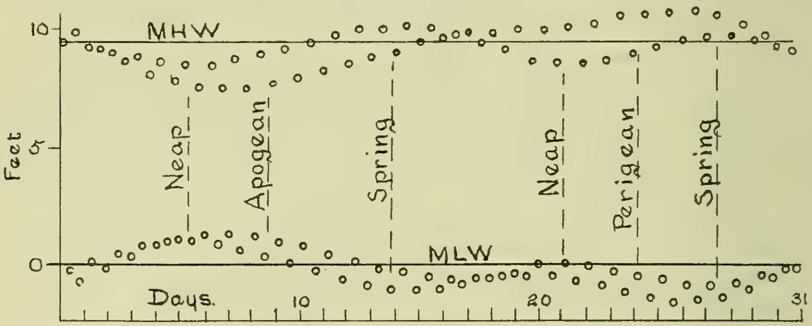

FIGURE 31.—Predicted high and low waters at Boston, Mass., January 1937.

the amplitude of the N_2 component exceeds that of the S_2 component. A typical monthly variation of high and low water at such a station is shown by the plot, in figure 31, of the predicted tides during January 1937, at Boston, Mass., where the amplitude of the N_2 component is the larger. These tides may be contrasted with the predicted tides at Sheerness, England, during the same month, shown in figure 32.

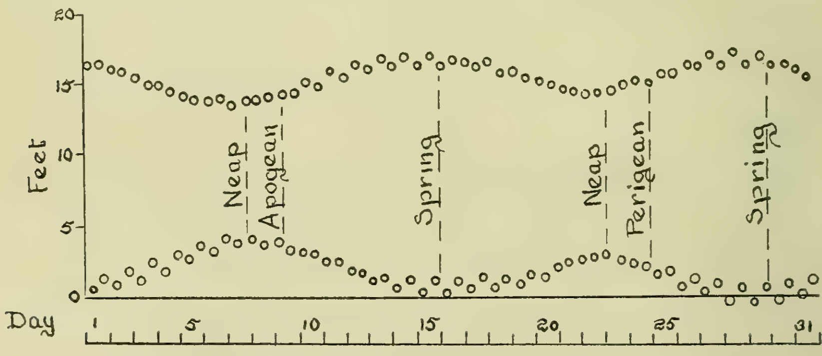

FIGURE 32.—Predicted high and low waters at Sheerness, England, January 1937.

At Sheerness the amplitude of the S_2 component considerably exceeds that of the N_2 component. The figures illustrate quite strikingly the reason why the terms "spring" and "neap" tides are commonly used in England, but not in the United States.

The tidal datum plane at Boston is mean low water, and the times of high and low water are on the standard time of the 75th meridian. At Sheerness the tidal-datum plane is mean low water of spring tides, and the time is Greenwich time.

The harmonic constants of the three principal semidiurnal components at these stations are taken as follows from table V, paragraph 134. The last two columns show the phase and parallax ages, computed from the epochs as indicated in equations (106) and (107).

Station	Amplitudes			Epochs			Ages in hours	
	M_2	S_2	N_2	$M_2°$	$S_2°$	$N_2°$	Phase	Parallax
Boston	4.371	0.699	0.995	330°	5°	300°	34.4	55.1
Sheerness	6.297	1.750	1.046	1°	56°	337°	54.1	44.1

The Greenwich times of the moon's phases, apogee and perigee, in January 1937 were, from the Nautical Almanac:

Moon's phases				Apogee	Perigee
Last quarter	Change	First quarter	Full		
Day Hour 4 2:22 p.m.	*Day Hour* 12 4:47 p.m.	*Day Hour* 19 2:02 p.m.	*Day Hour* 26 5:15 p.m.	*Day Hour* 6 3 p.m.	*Day Hour* 22 3 a.m.

The times of spring, neap, apogean and perigean tides at Sheerness are immediately determined from these astonomical data by adding the tidal ages. For Boston, they are similarly determined after correcting the times for the difference in longitude by subtracting 5 hours. The times of these tides then are:

Station	Spring tide	Neap tide	Apogean	Perigean
	Day Hour	*Day Hour*	*Day Hour*	*Day Hour*
Sheerness	14 11 p.m. 28 11 p.m.	6 9 p.m. 22 2 a.m.	8 11 a.m.	23 11 p.m.
Boston	13 10 p.m. 27 11 p.m.	5 8 p.m. 21 1 a.m.	8 5 p.m.	24 5 a.m.

These times are indicated in figures 32 and 33.

148. Exceptionally high and low waters are to be anticipated when the perigean and spring tides nearly coincide. Since the next succeeding apogean tide occurs one-half of an anomalistic month, or a little less than 14 days later, and the next succeeding spring tide one-half a synodic month, or a little more than 14 days later, it follows that when the tides after say the new moon are especially large, those after the next (or preceding) full moon are not.

As the length of the anomalistic month is approximately 27½ days while that of the synodic month is 29½ days, lunar perigee gains 2 days a month on the moon's phases. It follows therefore that perigean tides will most nearly coincide with spring tides at intervals of 7 months. Similarly, at stations having tides of the semidiurnal type, an exceptionally small tidal range is to be anticipated once during the month at 7 month intervals, occuring half way between the months at which exceptionally high and low waters occur.

EFFECT OF THE PRINCIPAL DIURNAL COMPONENTS

149. *The* K_1 *and* O_1 *components—tropic tides.*—Since the difference between the speeds of these components is relatively small, they combine to form a diurnal tidal fluctuation with an amplitude ranging from a minimum of K_1-O_1 to a maximum of K_1+O_1. The period from maximum to maximum is:

$$\begin{aligned}360°/(k_1-o_1) &= 360°/(15.041{,}068{,}6-13.943{,}035{,}6)\\ &= 360°/1.098033=327.859 \text{ hours.}\end{aligned}$$

This period is one-half of the tropical month (par. 62).

When the amplitide of the resultant of these two diurnal components is a minimum, the tides are called *equatorial tides*, since the moon is then near the Equator. When it is a maximum, the tides are called *tropic tides*, since this maximum results from the maximum inequality of the two daily tidal impulses, and therefore occurs when the moon has its greatest declination, near the celestial Tropics (par. 40).

150. *Effect of the diurnal components on high and low waters.*—Since the diurnal part of the tide rises once and falls once daily, it has a zero elevation (at mean sea level) at semidaily intervals approximating the period of the semidiurnal components. If the epochs of the K_1, O_1, and M_2 components are such that the resultant ordinate of the diurnal components is nearly zero at the two daily low waters of the M_2 component, the diurnal part of the tide evidently increases one of the two daily high waters and decreases the other, producing a diurnal inequality of the high waters, as may perhaps be seen more clearly by turning back to figure 13, page 22. Similarly, if these epochs are such that the diurnal part of the tide is nearly zero at the two daily high waters of the M_2 component, a diurnal inequality of the low waters is produced. Obviously, both the high and the low waters usually will show an inequality because of the diurnal components, but the inequality of the high waters is not, in general, the same as that of the low waters. As has been shown, these inequalities in the two daily tides vary from a minimum at the time of equatorial tides to a maximum at the time of tropic tides.

A characteristic fluctuation of tides of the mixed type is exemplified by the tide curve at San Francisco, Calif., at the time of a tropic tide, June 29, 1935, shown in figure 33.

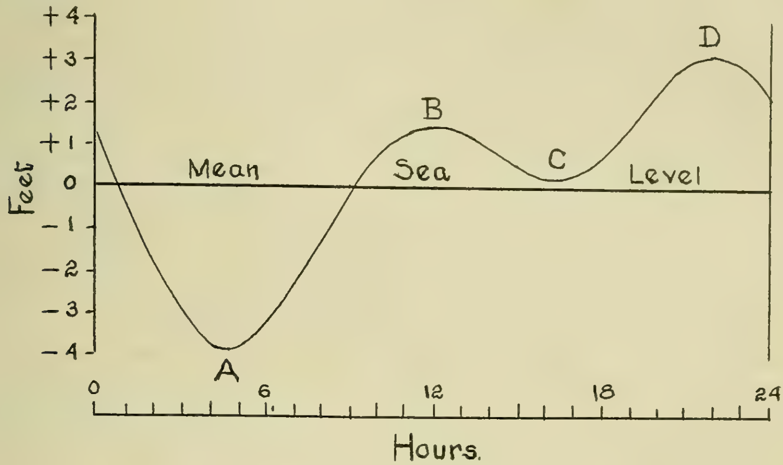

FIGURE 33.—Tropic tide, San Francisco, June 29, 1935.

A is the *lower low water* (LLW), *B* the *lower high water* (LHW), *C* the *higher low water* (HLW), and *D* the *higher high water* (HHW).

151. As shown in paragraph 68, the mean speed of the lunar diurnal part of the tide is m_1, the speed of the lunar day. It is therefore exactly one-half of the mean speed, m_2, of the lunar semidurnal part. Consequently, the resultant of the lunar diurnal components keeps in general step, from month to month, with the resultant of the lunar semidiurnal components. In most regions, the lunar components are so much larger than the solar that they determine the general shape of the daily tide curves. Usually, therefore, the higher and lower high and low waters at a tidal station always follow one another in the same *sequence*. If the higher high follows the lower low water, the lower high must follow the higher low, and vice versa, so that the sequence is established either as "HHW to LLW" or as "LLW to HHW." At San Francisco, for example, the sequence is HHW to LLW. At some stations, but exceptionally, the sequence changes during the year. Such a condition is to be anticipated when the principal solar diurnal component, P_1, is relatively large.

152. *Tropic and diurnal ranges, high- and low-water inequalities.*—The average difference, from month to month, in elevation between the higher high and the lower low waters of *tropic* tides is called the *great tropic range*, and the corresponding difference between the lower high and the higher low waters of tropic tides is called the *small tropic range*. The difference in the average heights of *all* higher high waters and the average heights of all of the lower low waters from day to day

for one or more tropical months is called the *great diurnal range,* or the *diurnal range.* The corresponding difference between the average heights of all of the lower high waters and the higher low waters is called the *small diurnal range.* These are all called *declinational* ranges, since they depend on the declination of the moon.

The diurnal high water inequality (DHQ) is defined as the difference between mean higher high water and mean high water. The *diurnal low water inequality* (DLQ) is similarly the difference between mean low water and mean lower low water. It is apparent that the great diurnal range is equal to the mean range plus (DHQ+DLQ) and the small diurnal range is equal to the mean range minus (DHQ+DLQ).

153. *Diurnal age.*—The *diurnal age* is the interval between the instant at which the moon is at its maximum monthly declination, either north or south of the Equator, and the time of tropic tides.

Since the K_1 and O_1 components of the equilibrium tides are in conjunction when the moon is at its maximum declination, the phases of these components of the actual tides then differ by the difference of their epochs and these components of the actual tides are in conjunction $(K_1{}^\circ - O_1{}^\circ)/(k_1 - o_1)$ hours later. Therefore:

$$\text{Diurnal age (in hours)} = (K^\circ{}_1 - O_1{}^\circ)/1.098 = 0.911(K_1{}^\circ - O_1{}^\circ) \quad (108)$$

For example, at Fort Hamilton, New York Harbor, $K_1{}^\circ = 104^\circ$ and $O_1{}^\circ = 98^\circ$. The diurnal age at this station is therefore

$$0.911\ (104\text{–}98) = 5.5 \text{ hours.}$$

The diurnal age at a station may amount to several days, and not infrequently is negative.

154. As was shown in paragraph 40, the amplitude of the semidiurnal part of the tide decreases as the declination of the moon increases, while that of the diurnal part increases with the declination. As a consequence the mean daily tidal range tends to decrease with the declination, but this decrease is overshadowed by the increasing range from lower low to higher high water. In tides of the semidiurnal type, the diurnal components do not obscure, to any marked degree, the spring and neap, perigean, and apogean variations due to the S_2 and N_2 components. In tides of the mixed type the variations in higher high and lower low waters, culminating twice a month in the tropic tides, become the outstanding characteristic, and obscure, more or less completely, the spring, neap, perigean, and apogean tides. In tides of the diurnal type, the diurnal components completely dominate the semidiurnal during a considerable part of the month. The fluctuations of the diurnal tides are, however, frequently so small that meteorological disturbances become their outstanding characteristic.

EFFECT OF OVERTIDES

155. Since the periods of the lunar overtides are one-half, one-third, and one-fourth of the period of the M_2 component, they unite with the latter to produce a tidal curve which is distorted from a sinusoidal curve, but is repeated without change in each successive period of that component. The form of the curve resulting from the combination of the M_2 and M_4 components at Philadelphia is shown in figure 34. At this station the amplitude of the M_2 component is 2.367 feet, and of the M_4 component 0.368 feet, their epochs being 49° and 7°, respectively.

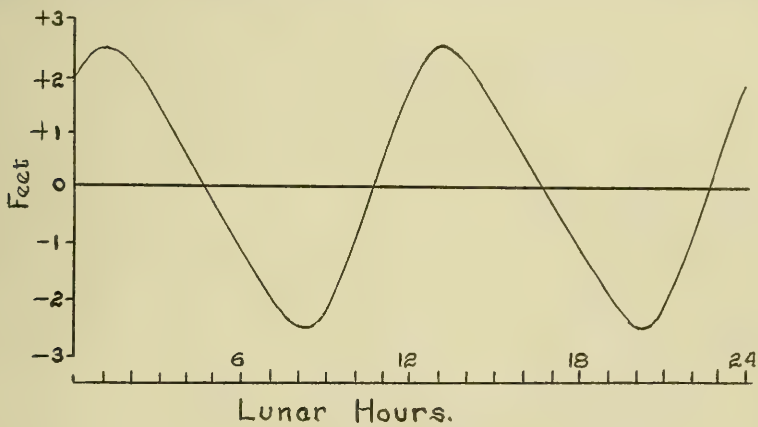

FIGURE 34.—Resultant of M_2 and M_4 components at Philadelphia.

The effect of the overtide in this case in increasing the interval from high water to low water and in decreasing the interval from low water to high water is apparent.

156. If a high water of the M_4 component nearly coincides with a high water of the M_2 component, the next high water of the overtide will nearly coincide with the low water of the primary component. The overtide will therefore raise the elevation of both the high and low waters with respect to mean sea level. Similarly if the epochs are such that a low water of the overtide nearly coincides with the high and low waters of the primary component, it will lower both the high and low waters of the resultant tide with respect to mean sea level. In the case illustrated in figure 34, the epochs differ by about 45° and the overtide has little effect in altering the relation of the high and low waters of the resultant with respect to mean sea level.

Chapter IV

TIDAL DATUM PLANES

157. *Principal tidal datums.*—The mean height of sea level, and the mean heights of low or high waters of various descriptions, afford the datums to which the elevations of upland areas, and of the bottom of the sea and of tidal waterways, ordinarily are referred. The datums which need be especially considered, and the abbreviations by which they are designated, are as follows:

Mean sea level, MSL.

Half tide level, HTL.

Mean low water and mean high water, MLW and MHW.

Mean lower low water and mean higher high water, LLW and HHW.

Mean low and high water of spring tides. In England, these datums are taken as mean low and high water of ordinary spring tides, after rejecting any spring tides which differ substantially from the usual, and are designated as LWOST and HWOST respectively.

In some cases channel depths at foreign ports are referred to mean high or low water of neap tides. Mean low and high waters of perigean, apogean, and tropic tides are rarely if ever used as a reference.

158. *Tidal ranges.*—The symbols conventionally assigned to the tidal ranges determined by the datums listed in the preceding paragraph, are as follows:

Mean range, Mn=MHW−MLW.

Diurnal or great diurnal range, Gt=HHW−LLW.

Spring range, Sg, mean low water to mean high water of spring tides.

MEAN SEA LEVEL

159. *Use.*—This datum is the basic plane of reference and the zero of the ordinates of the harmonic components of the tide. It is determined by averaging the observed hourly tidal heights, measured from a fixed bench mark, over a sufficient period of time. Because of the variation in the density of the waters of the oceans with changes in their temperature and salinity; because of the variation in the mean barometric pressure upon them; and because of the effect of winds, evaporation, and precipitation; mean sea level at different tidal stations may not be on precisely the same geodetic level surface. Thus mean sea level at Balboa, at the Pacific entrance to the Panama Canal, as determined from observations extending over 25 years, is nearly 0.7 foot higher than at Cristobal at the Atlantic entrance. In general, however, mean sea level at tidal stations which have a free connection with the sea, when determined from observations extending over a number of years, are so nearly on the same level surface that the difference between the elevation of any point on land above mean sea level at one station, as determined by a line of levels from that station, and the elevation of the same point above mean sea level at another station, is within the error inherent in long lines of levels. Mean sea level is therefore the standard reference datum for land elevations. At tidal stations on tidal rivers, or on land-locked bays and sounds with restricted entrances, the mean tidal height may be above mean sea level and is more correctly designated as *mean river* (*or bay*) *level.*

160. *Fluctuations in mean sea level.*—Small fortnightly, monthly, and semiannual fluctuations of mean sea level result from the long period harmonic components established by the attraction of the sun and moon (par. 71). These are, however, completely overshadowed by the disturbances resulting from storm tides, and smaller systematic meteorological disturbances.

161. *Storm tides.*—Occasional violent fluctuations of the water levels at a tidal station result from strong onshore or offshore winds. When these are of hurricane velocity, the water may be raised many feet. In the long run, storm disturbances raise (or lower) both the high and the low waters by substantially the same amount, and may be considered, therefore, as affecting primarily the heights of mean sea level.

162. *Systematic meteorological variations in mean sea level.*—Lesser atmospheric disturbances produce a less apparent, but more continuous, variation in mean sea level. The seasonal variations in the density of the water on the continental shelf and in the mean barometric pressure over wide areas of the oceans, with concurrent variations in the prevailing winds, and perhaps other meteorological causes,

result in fairly regular and consistent seasonal variations in the monthly mean sea level, even at stations not affected by the varying inflow from large rivers. A study of these variations at stations on the coasts of the United States, contained in Special Publication, No. 135, United States Coast and Geodetic Survey, shows that at North Atlantic ports mean sea level is quite consistently 0.2 feet or more higher during the summer months than during the winter months. The annual variation in mean sea level at the South Atlantic ports of Charleston and Fernandina approximates a foot, the highest elevations occurring in the fall. On the Gulf coast, the annual variation is about three fourths of a foot; and on the Pacific coast about half a foot. A comparison between the monthly changes in mean sea level at Portsmouth, N. H., and at Ketchikan, Alaska, and the monthly mean barometric pressure in the two regions, shows a striking correspondence in each case (Special Publications, No. 150 and 127, U. S. Coast and Geodetic Survey). At Balboa, at the Pacific entrance to the Panama Canal, an annual variation of about a foot in the elevation of the monthly mean sea level follows closely an annual variation of about 15° F. in the monthly mean water temperature. These seasonal variations are reflected in the values of the long-term components Sa and Ssa derived from harmonic analysis.

163. *Variations from year to year.*—Because of varying occurrence of storm tides from year to year, and the varying intensities of the causes of the seasonal variations in mean sea level, the mean annual sea level at a station varies from year to year. At stations on the coasts of the United States, where long-term observations have been made, these variations are, however, not often greater than 0.1 feet. The variations from year to year are quite uniform at all stations on the same sea coast.

Because of these small meteorological variations, mean sea level at a tidal station cannot be expected to be identically the same during any two periods, no matter how long these periods may be. A 9-year average is accepted by the United States Coast and Geodetic Survey as a primary determination which gives the elevation of mean sea level with sufficient accuracy for all practical purposes of that survey. Strictly speaking, however, it should be designated as the mean sea level during the particular period from which it was derived

HALF TIDE LEVEL

164. This datum is the elevation midway between mean low water and mean high water. Because of the distortion of the tide curves by the diurnal components and the lunar and solar overtides (par. 156), half tide level generally does not coincide exactly with mean sea level. On the Atlantic coast of the United States it usually is below

mean sea level, while on the Pacific coast, except in Alaska, it usually is above. Thus at Fort Hamilton, New York Harbor, it is 0.05 feet below mean sea level, at Philadelphia 0.16 feet below, and at San Francisco, 0.06 feet above that datum. Quite obviously, half tide level varies from month to month and from year to year by amounts that closely approximate the variations in mean sea level. Half tide level is rarely if ever used as a datum plane for land elevations and soundings, but affords a convenient reference for the correction of mean high and mean low waters.

LOW AND HIGH WATER DATUMS IN GENERAL

165. Since it clearly is desirable that the soundings on navigation charts, and the designated depths of improved channels, show the depths that generally can be counted on by navigators, they ordinarily are referred to one of the low water tidal datums, and not to mean sea level. Different low water datums are used for this purpose in different countries. The datums adopted in the United States are the most definitely determinable, but are not as low as those generally used in other countries. When comparing the channel depths in foreign ports with those in this country, the respective datums must be taken into consideration. Thus a channel 28 feet in depth at the adopted datum in a Canadian port might be 30 feet or more in depth if referred to the low water datum officially adopted in the United States for the region in which the harbor lies.

High water datums, while not suitable for charting, establish the tidal ranges, which are usually noted on charts to indicate the depths available at high water. In regions where the range between spring and neap tides is considerable, the elevation of neap high water is of especial importance, since it indicates the least depths at high water which can be counted on throughout the month.

166. *Low and high water datums do not establish a level surface.*—Obviously, as any of the several low and high water datums may be at a different height below or above mean sea level at different stations, these datums do not establish the same level surface from station to station, and are applicable only to the area in the vicinity of each station. Thus mean low water at the head of the Bay of Fundy is some 15 feet below the level of mean low water at the entrance to the bay. The change in the elevation of each of these datums, from station to station is, however, generally so gradual as to present no practical complications.

167. *Meteorological variations in high and low water datums.*—The variations in mean sea level from month to month, and from year to year, produce nearly identical variations in the several low and high water datums. They do not, however, produce any substantial

variation in the height of these datums with respect to mean sea level or to half tide level. The variations peculiar to the low and high water datums, and the corrections to be made therefor, may therefore be determined by taking the height of these datums with respect to mean sea level, or half tide level, during the period of the observations. The corrected heights, subtracted from or added to the elevation of mean sea level, give them the corrected elevations of the datums.

168. *Changes caused by channel improvements.*—A major enlargement of a tidal waterway, by dredging for the improvement of its navigability, may change materially the elevations of the low and high water datums along it. A classic example is the effect of the improvement of the Clyde in Scotland, which during the last century was converted from a shallow stream, fordable at low tide, to a waterway for deep-draft vessels. The enlargement lowered the low water levels at Glasgow by more than 8 feet, raised the high water levels by some 2 feet, and consequently lowered the midtide level by 3 feet. In exceptional cases, the low water datum may even be raised by channel enlargement. Extensive channel improvements may therefore require a revision of established tidal datum planes.

MEAN LOW AND MEAN HIGH WATER

169. *Definition.*—Mean low water is, as its name implies, the average height of all low waters over a long period of time, and mean high water is the average height of all high waters. Because of variations in the heights of high and low waters between springs and neaps in the half synodic month, between perigean and apogean tides in the anomalistic month, and between tropic and equatorial tides in the half tropic month, a determination of mean high or low water, with respect to mean sea level, from observations extending over a day or a week, might differ quite widely from the long time mean. To eliminate these variations the tides must be averaged over a period in which these variations go through almost, if not quite, their entire range one or more times. For a determination of mean low or mean high water, the shortest period suitable for this purpose is 29 days. This period, which is sometimes called a *lunation*, is so close to the synodic month of 29½ days as to practically eliminate the spring and neap variations. It is sufficiently close to the anomalistic month of 27 days, 13 hours, to nearly eliminate the perigean variation, and to the tropical month of 27 days, 8 hours, to nearly eliminate the declinational variation. Longer periods theoretically should be multiples of 29 days. For convenience of computation it is more usual to take successive 29-day periods, beginning say on the first of each

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month. If the observations extend over a year, no sensible error is introduced by taking all of the low or high waters for the 365 or 366 days.

170. *Use.*—Mean low water datum is the most readily determined of the several low water planes, and adequately serves as a plane of reference for navigation charts and for the designation of channel depths when the tidal range is moderate. It is the official reference plane for navigation charts and for federally improved channels on the Atlantic and Gulf coasts of the United States. Obviously low water of the varying tides is as often as not below this datum. At Eastport, Maine, where the mean tidal range is 18.2 feet, the normal tide occasionally falls 3 feet or more below mean low water datum; but at most of the other stations on the Atlantic coast of the United States, where the tidal range is much less, such *minus tides* (except those due to storms) do not often exceed 1 or 2 feet, and ordinarily are less.

171. *Correction for longitude of the moon's node.*—Because of the variation in the amplitudes of the tidal components with the changing inclination of the moon's orbit to the Equator (par. 102), the several tidal ranges and high and low water datums go through a small variation in a period of 19 years. The mean range derived from observations extending over a month or a year may be reduced to its true mean value by applying a reduction factor, conventionally designated as F(Mn). The corrected mean low water datum is then found by subtracting one-half of the corrected mean range from half tide level; and the corrected mean high water datum by the corresponding addition. These corrections are called the corrections for the longitude of the moon's node, since this longitude determines the inclination of the moon's orbit.

172. The numerical values of the reduction factor F(Mn) are derived by deducing an expression for the mean range, Mn, in terms of the amplitudes of the tidal components, and applying to these amplitudes the reduction factors derived in paragraphs 124–126, determined by the inclination, I, of the moon's orbit during the period of the observations. To simplify the correction, the amplitudes of the semidiurnal components are assumed to be proportional to the mean values of the coefficients of their equilibrium components, given in table IV, paragraph 129; and the amplitudes of the diurnal components also proportional to the mean values of the corresponding coefficients. The relation between the amplitudes of the diurnal and semidiurnal components is established by the ratio, $(K_1+O_1)/M_2$, of these actual components at the station, as determined by harmonic analysis, or inferred from available data. The derivation of these factors is explained at length in appendix II.

173. The accepted values of F(Mn) corresponding to successive values of I, and of $(K_1+O_1)/M_2$ ranging from 0 to 1, are shown in table VI. This table is abstracted from table 14 of the Manual of Tides by Harris, published in the Report of the United States Coast and Geodetic Survey for 1894, part II.

TABLE VI.—*Factor F(Mn) for correction of* Mn *for longitude of the moon's node*

$I=$	18°.5	19°	20°	21°	22°	23°	24°	25°	26°	27°	28°	28°.5
$\frac{K_1+O_1}{M_2}=0.0$	0.970	0.972	0.977	0.982	0.988	0.994	1.000	1.006	1.012	1.019	1.026	1.029
.2	.971	.973	.978	.982	.988	.994	1.000	1.006	1.012	1.019	1.026	1.029
.4	.972	.974	.979	.983	.988	.994	1.000	1.006	1.012	1.018	1.025	1.028
.6	.974	.976	.980	.984	.989	.994	1.000	1.006	1.011	1.017	1.023	1.026
.8	.977	.979	.982	.986	.990	.995	1.000	1.005	1.010	1.015	1.020	1.022
1.0	.980	.982	.985	.989	.992	.996	1.000	1.004	1.008	1.013	1.017	1.019

174. The values of I at the middle of each calendar year from 1890 to 1969 are shown in table VII.

TABLE VII.—*Inclination, I, of moon's orbit at middle of year*

Year	0	1	2	3	4	5	6	7	8	9
189—	24°.6	26°.1	27°.4	28°.2	28°.6	28°.5	27°.9	26°.9	25°.5	23°.9
190—	22°.2	20°.5	19°.2	18°.4	18°.4	19°.0	20°.2	21°.8	23°.6	25°.2
191—	26°.6	27°.7	28°.4	28°.6	28°.3	27°.6	26°.4	24°.9	23°.2	21°.5
192—	20°.0	18°.8	18°.3	18°.5	19°.4	20°.8	22°.5	24°.2	25°.8	27°.1
193—	28°.0	28°.5	28°.5	28°.1	27°.2	25°.9	24°.3	22°.6	20°.9	19°.5
194—	18°.6	18°.3	18°.8	19°.9	21°.5	23°.2	24°.9	26°.3	27°.5	28°.3
195—	28°.6	28°.4	27°.8	26°.7	25°.3	23°.6	21°.9	20°.3	19°.0	18°.4
196—	18°.4	19°.2	20°.5	22°.1	23°.8	25°.5	26°.2	27°.9	28°.5	28°.6

175. *Application of corrections.*—The correction to the observed tidal range in any month or year is readily determined from tables VI and VII if the harmonic constants M_2, K_1, and O_1 have been determined for the station. If these constants are not available, the ratio of $(K_1+O_1)/M_2$ at the nearest station for which they are determined may be used, if the tide is of a similar type. Otherwise the ratio is taken as equal to 2 (DHQ+DLQ)/Mn, as explained in appendix II.

176. *Example.*—At Fort Hamilton, New York Harbor, $K_1=0.322$ feet, $O_1=0.172$ feet, and $M_2=2.210$ feet. The value of $(K_1+O_1)/M_2$ is then 0.22. For comparison the great diurnal range is 5.29 feet and the mean range is 4.73 feet. The value of 2(DHQ+DLQ)/Mn is then 0.24. The mean tidal range for 1902 was 4.79 feet. The value of I for 1902 is, from table VII, 19°.2. The corresponding value of F(Mn) for $(K_1+O_1)/M_2=0.2$ is, from table VI, 0.974. The mean range corrected for the longitude of the moon's node is then $4.79\times0.973=4.67$; which is 0.12 feet less than the observed range. Observed mean low water for the year was 2.44 feet below mean sea level, and mean high water 2.35 above mean sea level. Applying half the correction to each, these elevations, corrected for the longitude of the moon's node, become respectively 2.38 feet and 2.29 feet.

177. The annual mean tidal ranges at Fort Hamilton, as observed, and after correction for the longitude of the moon's node, for the years 1893 to 1932 are plotted in figure 35 from the data given in Special Publication, No. 111 of the United States Coast and Geodetic Survey (1935 edition).

The variation in the observed annual ranges with the longitude of the moon's node, N, is apparent in the figure. The increase in the

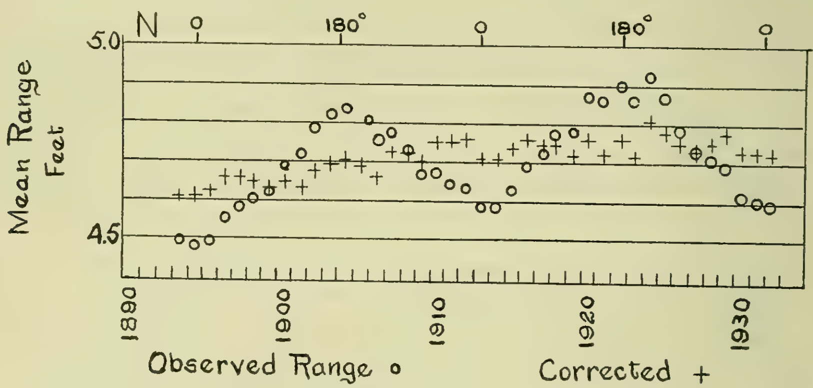

FIGURE 35.—Observed and corrected annual mean tidal range Fort Hamilton, New York, Harbor.

corrected ranges between 1902 and 1912 coincides with the major enlargement of the harbor entrance in the dredging of the Ambrose Channel during this period.

178. Table VI is not extended to give the values of F(Mn) for values of $(K_1+O_1)/M_2$ in excess of unity. When this ratio exceeds unity the tides are decidedly of the mixed type, and the diurnal inequalities become their important feature. A study included in Special Publication, No.115, United States Coast and Geodetic Survey, of the annual mean tidal ranges at San Francisco, Calif., where the ratio $(K_1+O_1)/M_2$ is 1.1, shows that any effects of the longitude of the moon's node on the annual mean tidal ranges during the 26-year period from 1898 to 1923 are completely overshadowed by accidental variations in the ranges, and that the correction of the observed ranges for the longitude of the moon's node serves little purpose in reducing the observations to better concordance.

179. The correction for the longitude of the moon's node is applied only in an independent determination of mean low and high water datums at a station. Ordinarily these datums are determined by a comparison of the high and low waters with those at an established base station (par. 196 et seq.) at which the correction already has been applied. The approximations inherent to this correction are practically eliminated when the datums are determined from observations extending over nine years, the period in which the moon's node

retrogresses through substantially 180°. The corrections are of importance in a close study of the effect of a channel improvement upon the mean tidal ranges.

180. *Precision of observations.*—Observations extending over 9 years are considered by the United States Coast and Geodetic Survey to afford a primary determination of the mean low and high water datums at a tidal station. In general observations for a year, corrected for the longitude of the moon's node, determine the relation of these datums to mean sea level, or half tide level, at the station, within 0.05 foot of the 9 year mean; and observations for a month within 0.1 foot (Special Publication 135, U. S. Coast and Geodetic Survey, p. 107).

MEAN LOW AND HIGH WATERS OF SPRING TIDES

181. *Differing definitions.*—Spring low waters and high waters are most accurately defined as the low waters and high waters nearest the time of conjunction of the principal lunar and solar semidiurnal components of the tide, M_2 and S_2 (par. 143); but may be more loosely taken as the lowest low waters and highest high waters occurring semimonthly soon after new and full moon. At English ports and in other regions where the lowest low waters and highest high waters follow consistently the conjunction of these components, and the tides run through a regular variation from springs to neaps, with small diurnal differences, spring tides are readily identified in the recorded low and high waters, and their means over a number of months afford fairly definite datums. Because of the small number of spring tides in a half year or a year, a single abnormal tide would have a relatively large effect on the mean value. Thus a storm disturbance of 3 feet would change the mean low water of spring tides during a 6-month period by a quarter of a foot, while it would change the mean of all low waters during the same period by less than one hundredth of a foot. It is therefore the English practise to reject abnormal spring tides from the computations, and to designate the datums as mean low and high water of ordinary spring tides. These datums depend to some extent, consequently, on the judgment of the computer.

182. In regions where the tides have a considerable variation from apogee to perigee, or from equatorial to tropic tides, the spring tides may not be as readily identified; and when the diurnal components of the tide are large, the low and high waters occurring next before the time of conjunction of the M_2 and S_2 components may differ widely from those next following. In some countries the datums designated as mean low and high waters of spring tides are in fact the means of the lowest low and highest high waters which occur soon after the successive full and new moons. Such determinations obviously are somewhat haphazard, but afford a low water datum below which the

tide does not often fall. It might be better named the mean lower low water of spring tides.

183. In the United States, mean low water of spring tides is taken by the Coast and Geodetic Survey as the average of the two low waters nearest the successive times of spring tides, these times being determined by adding the phase age (par. 144) to the hour of new or full moon. The variations in the diurnal inequality at the times of spring tide are thereby eliminated. With this procedure four tidal heights per month enter the computation both of mean low water and of mean high water of spring tides; but observations must extend over a long period to afford a mean in which the other systematic and the accidental variations in the tide are satisfactorily eliminated.

184. *Approximate values.*—It is shown in appendix II that the height of mean high water above half-tide level, and of mean low water below half-tide level is equal to the amplitude of the M_2 component increased by a relatively small correction due to the displacement by the other components of the time of high and low water. The S_2 component does not have, therefore, any large effect on the elevation of mean high or low water or the mean tidal range. At the time of spring tides, however, the S_2 component is in conjunction with the M_2 component, and the height of high water is increased, and of low water decreased, by substantially its amplitude. It follows therefore that mean high water of spring tides, as the term is used in the United States, is closely approximated by adding the value of S_2, as computed by harmonic analysis, to the corrected elevation of mean high water, as determined by observation; and mean low water of spring tides by subtracting this value from the established mean low water. In other words:

$$\mathrm{LWOST} = \mathrm{MLW} - S_2 \tag{109}$$

$$\mathrm{HWOST} = \mathrm{MHW} + S_2 \tag{110}$$

$$\mathrm{Sg} = \mathrm{Mn} + 2S_2 \tag{111}$$

For example, the elevation of mean low water below half-tide level at Fort Hamilton, New York Harbor, is determined from observations extending over a long period, to be 2.37 feet, and the value of S_2 at this station is 0.44. The elevation of low water of spring tides, from equation (109) is then 2.81 below half-tide level. Its value computed directly from observations extending over several years is 2.79. Similarly at the Presidio of San Francisco, the elevation of low water of spring tides from equation (109) is 2.37 feet below half-tide level, while its elevation from direct observation, is 2.36 feet. The correspondence is, therefore, very close at these stations.

If then the harmonic components at a station have been computed, and a good determination made of the mean low water datum, formula (109) generally affords a satisfactory determination of the mean low water of spring tides. After a satisfactory determination of mean low water of spring tides has been made at one station, that at other stations in the vicinity may be derived by comparison (par. 202).

185. *Use.*—Mean low water of ordinary spring tides is the reference plane for the British Admiralty charts and generally for works of harbor improvement in the British Empire. It is used in some other countries as well. In Canada, low water datum is taken as from 0.5 to 1.5 feet below the mean of the lowest low waters of spring tides. In the United States, mean low water of spring tides is used as a datum by the Coast and Geodetic Survey only on the Pacific coast of the Panama Canal Zone, where the range from springs to neaps is marked and regular.

MEAN HIGH AND LOW WATERS OF NEAP, PERIGEAN, APOGEAN AND TROPIC TIDES

186. These datums are determined in the same manner as the high and low waters of spring tides. Thus the mean high water of neap tides is taken as the mean of the successive pairs of high waters nearest the time of neap tides, and is approximately equal to $MHW-S_2$, the neap range being approximately equal to $Mn-2S_2$. Mean high water of perigean tide is similarly the mean of the successive pairs of high waters nearest the time of perigean tide, as determined by adding the parallax age to the time of lunar perigee. It is approximately equal to $MHW+N_2$, while mean high water of apogean tide is approximately equal to $MHW-N_2$. The lower low, higher high, higher low, and lower high waters of tropic tides are the averages of the lower low, higher high, higher low, and lower high waters at the time of tropic tides as derived from the diurnal age. As has been stated, these datums are rarely if ever used as reference planes for charts. The elevations of mean high and low waters of neap tides, are however of importance at stations having a marked and regular range from springs to neaps, and especially at ports where navigation is on the tide.

MEAN LOWER LOW AND HIGHER HIGH WATERS

187. These planes are sometimes called declinational planes, since the lower low and higher high waters vary with the declination of the moon and sun. Mean lower low water is the average height of the lower of the two daily low waters of tides of the semidiurnal and mixed types. Since the lunar day is longer than the calendar day, occasion-

ally but one low water occurs (about noon) during the calendar day even when the tide is wholly semidiurnal. It is included in, or excluded from, the summation according to its relation to the preceding low water. If two low waters of the same height occur on a calendar day, but one is included. When, however, the tide becomes temporarily dirunal, each low water is included in the summation. Mean higher high water is similarly computed.

188. *Use.*—Where the tides are of the mixed type, mean lower low water affords a more suitable reference plane than mean low water, and is the official reference plane for navigation charts and channel improvements on the Pacific coast of the United States. While this datum is below mean low water (by as much as 1.8 feet at Seattle) yet one of the two daily tides is as like as not to fall below it, sometimes considerably. Thus at Seattle normal tides occasionally fall as much as 3 feet below mean lower low water.

At localities having a tide which is wholly diurnal, mean low water and mean lower low water become synonomous. On the Gulf of Mexico, where the tides are generally of the diurnal type, but small and irregular, mean low water affords a more satisfactory reference plane than mean lower low water, and is the officially adopted plane in the United States.

189. *Corrections to short term determinations.*—An independent determination of mean lower low or higher high water at a station, like that of mean low or high water, must extend over a minimum period of 29 days to eliminate the monthly variations in the tidal range. Furthermore, the elevations of mean lower low and higher high waters vary with the changing declination of the sun from month to month during the year, as well as varying with the changing inclination of the moon's orbit during a period of 19 years. The corrections to reduce to their true mean values, determinations based on observations during a month or a year, are derived by applying a reduction factor, conventionally designated 1.02 F_1, to the diurnal low and high water inequalities, DLQ and DHQ (par. 152). The corrected diurnal low water inequality is then subtracted from the corrected mean low water datum, derived as explained in paragraphs 171 to 177; and the corrected diurnal high water inequality added to the corrected mean high water datum.

190. The derivation of the reduction factors, 1.02 F_1, is explained in appendix II. The computed values for each month of the year from 1891 to 1950, are given in table 7, Special Publication No. 135, United States Coast and Geodetic Survey (Tidal Datum Planes), pages 114–115. The values from 1921 to 1950 are extracted therefrom in the following table:

TABLE VIII.—*Factors 1.02 F_1, for correcting diurnal inequality to mean value*

Year	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.	Mean
1921	0.99	1.18	1.42	1.31	1.06	0.95	0.99	1.21	1.46	1.32	1.07	0.96	1.160
1922	1.01	1.22	1.46	1.34	1.08	.96	1.00	1.22	1.47	1.33	1.07	.96	1.177
1923	1.01	1.22	1.46	1.34	1.08	.96	1.00	1.20	1.46	1.31	1.06	.95	1.171
1924	1.00	1.20	1.42	1.30	1.05	.94	.98	1.17	1.39	1.26	1.02	.92	1.138
1925	.97	1.14	1.35	1.24	1.00	.90	.94	1.11	1.31	1.19	.98	.88	1.084
1926	.92	1.08	1.26	1.17	.96	.87	.89	1.06	1.23	1.13	.93	.85	1.029
1927	.88	1.03	1.19	1.10	.91	.83	.86	1.00	1.16	1.07	.89	.81	.978
1928	.85	.99	1.12	1.05	.88	.80	.83	.96	1.11	1.02	.86	.79	.938
1929	.82	.95	1.08	1.01	.85	.78	.80	.93	1.07	.99	.84	.77	.908
1930	.80	.92	1.05	.99	.83	.76	.79	.91	1.04	.97	.82	.76	.887
1931	.79	.91	1.03	.97	.82	.76	.78	.91	1.04	.96	.82	.76	.879
1932	.79	.91	1.03	.97	.82	.76	.78	.91	1.04	.96	.82	.76	.879
1933	.79	.91	1.04	.98	.83	.76	.79	.92	1.05	.98	.83	.77	.888
1934	.80	.92	1.06	1.00	.84	.78	.80	.94	1.08	1.01	.85	.78	.905
1935	.82	.95	1.10	1.03	.87	.80	.83	.97	1.13	1.05	.88	.81	.937
1936	.85	1.00	1.15	1.08	.90	.83	.86	1.01	1.19	1.10	.92	.84	.978
1937	.88	1.04	1.22	1.14	.94	.86	.90	1.07	1.26	1.16	.96	.88	1.026
1938	.92	1.09	1.30	1.21	.99	.90	.94	1.13	1.34	1.23	1.01	.92	1.082
1939	.97	1.15	1.38	1.28	1.04	.94	.98	1.18	1.42	1.29	1.05	.95	1.136
1940	1.00	1.21	1.44	1.33	1.07	.96	1.00	1.22	1.47	1.33	1.07	.96	1.172
1941	1.01	1.22	1.46	1.34	1.07	.96	1.00	1.22	1.47	1.33	1.07	.96	1.176
1942	1.01	1.21	1.45	1.33	1.07	.96	.99	1.20	1.43	1.30	1.04	.94	1.161
1943	.98	1.18	1.39	1.28	1.03	.93	.96	1.15	1.36	1.24	1.00	.91	1.118
1944	.95	1.13	1.31	1.21	.99	.89	.92	1.09	1.29	1.17	.96	.87	1.065
1945	.90	1.06	1.23	1.14	.94	.85	.88	1.04	1.20	1.10	.91	.83	1.007
1946	.87	1.01	1.16	1.08	.90	.82	.84	.99	1.14	1.05	.88	.80	.962
1947	.83	.97	1.11	1.03	.87	.79	.82	.95	1.09	1.01	.85	.78	.925
1948	.81	.94	1.07	1.00	.84	.77	.80	.93	1.06	.98	.83	.76	.899
1949	.79	.91	1.04	.98	.83	.76	.78	.91	1.04	.96	.82	.76	.882
1950	.79	.90	1.03	.97	.82	.76	.78	.91	1.04	.96	.82	.76	.878

The mean annual values of the correction, given in the last column, are the reduction factors to be applied the diurnal inequalities derived from observations extending through the year.

The approximations introduced by these corrections are practically eliminated in a determination of these datums from observations extending over 9 years.

191. *Example.*—The application of the reduction factors to obtain the corrected mean lower low and mean higher high waters is illustrated by the determination of these corrections to the observed annual mean tidal heights above an arbitrary datum plane at Ketchikan, Alaska, in 1922, given in Special Publication, No. 127, United States Coast and Geodetic Survey (Tides and Currents in Southeast Alaska).

The observed heights are:

Mean higher high water (HHW)=21.55.
Mean high water (MHW)=20.75.
Mean lower low water (LLW)=6.06.
Mean low water (MLW)=7.43.
Annual half tide level (HTL)=½ (20.75+7.43)=14.09.
Mean high water above HTL=6.66.
Mean low water below HTL=6.66.
Mean range (Mn)=20.75−7.43=13.32.
Annual high water inequality (DHQ)=21.55−20.75=0.80.
Annual low water inequality (DLQ)=7.43−6.06=1.37.

From table V, paragraph 134:

$K_1=1.648 \qquad O_1=1.014 \qquad M_2=6.138$

Whence $(K_1+O_1)/M_2=0.43$.

From table VII, paragraph 174, $I=18°.3$.

From table VI, paragraph 173, $F(\mathrm{Mn})=0.971$.

Corrected $\mathrm{Mn}=13.32\times0.971=12.93$.

Correction to MHW and MLW $=\frac{1}{2}(13.32-12.93)=0.20$.

Corrected MHW above HTL $=6.66-0.20=6.46$.

Corrected MLW below HTL $=6.46$.

$1.02\ F_1$ (from table VIII) $=1.177$.

Corrected DHQ $=0.80\times1.177=0.94$.

Corrected DLQ $=1.38\times1.177=1.62$.

Corrected HHW above HTL $=6.46+0.94=7.40$.

Corrected LLW below HTL $=6.46+1.61=8.08$.

Corrected HHW on staff $=14.09+7.40=21.49$.

Corrected LLW $=14.09-8.08=6.01$.

It may be noted that in this case the corrections to DHQ and DLQ nearly counterbalance the corrections to Mn. The correction factor $1.02\ F_1$ to the mean annual diurnal inequalities decreases with I, while the correction factor $F(\mathrm{Mn})$ to the mean range increases with that angle. A glance at table VIII shows, however, that the plane of lower low water goes through marked variations from month to month.

192. *Precision of determinations.*—As with the other datum planes, a determination of mean lower low or higher high water from corrected observations extending over a period of 9 years is considered by the Coast and Geodetic Survey as a primary determination. In general, observations for a year, similarly corrected, determine the relation of these datums to mean sea level, or half tide level, within 0.1 foot of the 9 year determination; and observations over a month with a quarter of a foot. At least 3 days observations should be used to determine this datum within a foot of the long term value. (Special Publication 135, U. S. Coast and Geodetic Survey, p. 124.)

OTHER DATUM PLANES

193. *Harmonic tide plane.*—A tidal plane often referred to, and used at some ports in India, is that at an elevation of $M_2+S_2+K_1+O_1$ below mean sea level. It nearly coincides with what might be called tropic lower low water of spring tides. It has the advantage of being so low that normal tides rarely fall below it.

194. *Arbitrary datum planes.*—As will later be shown, the tidal datums herein before listed, after being determined from a more or less extended set of observations, are referred to standard bench marks which thereafter become the controlling reference for charts, tide tables, and channel depths. In some countries local datum planes,

established by reference to such a bench mark, are arbitrarily adopted for these purposes, without particular relation to one of the characteristic tidal planes.

TYPICAL RELATIONS BETWEEN DATUM PLANES

195. The relations between these planes at Fort Hamilton, New York Harbor, where the tide is of the semidiurnal type, and at the Presidio, San Francisco Harbor, where the tide is of the mixed type, are as follows:

	Elevations below mean sea level (feet)	
	New York (1912–30)	*San Francisco (1898–1923)*
Mean low water	*2. 42*	1. 87
Mean lower low water	2. 64	*3. 02*
Low water of spring tides	2. 88	2. 26
Harmonic tide plane	3. 20	4. 14

DETERMINATION OF TIDAL DATUMS BY COMPARISON

196. Because of the variation from day to day, from month to month, and from year to year in the elevation of mean sea level, and the periodic variations in the height of the successive high and low waters with respect to mean sea level, long-continued observations are necessary to establish, with good precision, the several tidal datums at a station; but after these datums have been established at one primary or base station they may be determined at other stations in the same region, where the tidal variations are due to like causes, by comparing, during a relatively short period, the high and low water elevations at the secondary station with those at the base station. This method is applicable only when the tides at the base and secondary stations are similar; i. e. when the ratio 2 (DHQ+DLQ)/Mn at the two stations is substantially the same, and the higher high and lower low waters are in the same sequence (par. 151). Such conditions are to be anticipated at stations on the same general embayment of the coast line, and with free connections with the sea. They may be fulfilled at stations several hundred or even a thousand miles apart. The method of comparison is not applicable to stations on tidal rivers and estuaries in which the water levels are sensibly affected by the inflow from large rivers.

197. *Establishment of half-tide level by comparison.*—While the violent disturbances produced by storms may vary considerably even at stations in the same bay, the ordinary fluctuations of mean sea level, and of half-tide level, when averaged over a sufficient period of days, generally affect the elevations of these datums by substantially the same amount over quite extensive areas. To determine the half-tide level at a secondary station from the established datum at a base station, concurrent observations are therefore made for a suitable

period of the heights of high and low waters above arbitrarily selected zero elevations at the two stations. These heights are called the respective high and low waters on the staff. The mean high and low waters, and the half-tide levels on the staff at the two stations during the period of observation are computed. The difference between the established half-tide level and the observed half-tide level at the primary or base station gives the correction to be applied to the observed half-tide level at the secondary station.

198. *Mean sea level by comparison.*—If the base and secondary stations both have a free connection with the sea or are freely connected with each other by deep water, so that they both may be presumed to have the same overtides, the difference between mean sea level and half-tide level should be the same at both. This difference, as determined at the base station, applied to the corrected half-tide level at the secondary station, gives mean-tide level at the secondary station.

199. *Mean high and low waters by comparison.*—While the tidal range often varies materially from station to station in the same region, the heights of the successive low and high waters with respect to half-tide level at one station are proportional to those at another if the amplitudes of the components of the tide at one of the two stations have a constant ratio to those at the other, and the epochs of the several components at one station differ from those at the other by a constant angle. These conditions are to be expected when the tides at the two stations are both produced by the same offshore fluctuations of the ocean. They are exemplified by the relationship of the principal tidal components at stations on the New England coast north of Cape Cod. Harmonic constants have been determined at Portland, Maine, at Pulpit Harbor, 80 miles to the northeast, at Eastport, 190 miles northeast, and at Boston, 90 miles to the south of Portland. The ratio of the amplitudes of the principal tidal components at these stations to those at Portland, and the difference in the epochs of the respective components, are shown in the following tabulation, prepared from the data set forth in table V, paragraph 134. To extend the comparison, the ratios of the amplitudes of the principal components at Fernandina, Fla., 1,000 miles to the southward, to those at Portland are added, together with the difference in their epochs.

	Pulpit Harbor		Eastport		Boston		Fernandina	
	H/H_0	$\kappa-\kappa_0$	H/H_0	$\kappa-\kappa_0$	H/H_0	$\kappa-\kappa_0$	H/H_0	$\kappa-\kappa_0$
M_2	1.12	−4°	1.96	2°	1.00	6°	0.65	−96°
S_2	1.11	−5°	2.00	6°	1.00	5°	.73	−73°
N_2	1.11	−4°	1.82	6°	1.05	8°	.62	−79°
K_1	.99	−3°	1.04	−3°	.97	2°	.74	−5°
O_1	1.03	−3°	1.07	0°	.99	6°	.71	+18°

It may be noted that the ratios of the amplitudes of the semidiurnal components at Eastport to those at Portland are quite consistent, as are the ratios of the diurnal components, but the ratios of the semidiurnal differ widely from the diurnal. At Fernandina a similar divergence occurs in the differences of the epochs.

200. Since the successive heights of high waters and low waters with respect to half-tide level at one station are found to have a substantially constant ratio to these heights at another station in the same region, the long-term means of the high and low waters at the two stations are proportional to the respective mean values during any period of concurrent observations. The ratio of the mean range at the primary station during a period of concurrent observations to its established long-term mean, applied to the mean range during the same period at the secondary station, gives therefore the corrected mean range at the secondary station. The corrected heights of mean high water and mean low water on the staff are then obtained by adding and subtracting one-half of the corrected mean range to the corrected height of half tide on the staff.

201. It may be observed that a comparison, if based on a fairly long set of concurrent observations, will give reliable results even when the timing of the components is not the same at the two stations, for, as shown in appendix II, mean range at each station depends on the M_2 component and the ratios of the other components thereto, and not on the epochs of these components.

202. *Mean low water and mean high water of spring tides by comparison.*—It has been shown (par. 184) that the spring range may be taken as $Mn+2S_2$. Since the amplitude, S_2 ordinarily has a constant ratio to the amplitudes of the other principal components at stations in the same region, and hence to the respective mean ranges at these stations, the ratio of the spring range to the mean range should be the same at all such stations. After this ratio has been determined at a base station, it may be applied to the corrected mean range at any secondary station, as derived by comparison, to determine the spring range at the secondary station. Mean low water of spring tides at the secondary station is then one-half the spring range below the corrected half-tide level, and mean high water of spring tides one-half of the spring range above the corrected half-tide level. At stations on the Pacific coast of the Panama Canal Zone, for example, the ratio of spring range to mean range is 1.26, and the elevation of low water of spring tides is taken as HTL$-$0.63 Mn.

203. *Mean lower low and mean higher high waters by comparison.*—It has been seen that the elevation of mean higher high water exceeds that of mean high water by the diurnal high water inequality, DHQ, and the elevation of mean lower low water is that of mean low water less the diurnal low water inequality, DLQ. The elevation of mean

high water and of mean low water depends principally on the semidiurnal components of the tides, while the diurnal inequalities depend wholly on the diurnal components. Since the ratio of the amplitudes of the diurnal components at two stations tends to differ from the ratio of the amplitudes of the semidiurnal components, even when the two stations are in the same region, the diurnal inequalities are separately compared. The ratios of the observed mean inequalities at the base station, during a period of concurrent observations, to the established long term mean values of these inequalities at the base station, applied to the observed mean inequalities at the secondary station, give the corrected values of the inequalities at the secondary station. These, added to and subtracted from the corrected mean high and low waters on the staff, give the corrected mean higher high and lower low waters on the staff at the secondary station.

204. *Example.*—The computation of the mean lower low water datum at Anacortes, Wash., from concurrent observations extending over 7 days at this station and at a base station at Seattle is shown below. The "accepted datums" in the second column are the established long-term means at Seattle.

	Seattle			Anacortes	
	Observed	Accepted	Ratio or difference	Observed	Corrected
HHW	18.40	18.74	------------	22.94	------------
MHW	17.84	17.88	------------	22.47	------------
MLW	10.35	10.24	------------	17.80	------------
LLW	7.86	7.41	------------	15.57	------------
Mn	7.49	7.64	1.020	4.67	------------
½Mn	3.75	3.82	1.020	2.335	2.38
HTL	14.10	14.06	−.04	20.14	20.10
DLQ	2.49	2.83	1.137	2.23	2.54

Mean lower low water at Anacortes, corrected = HTL − ½Mn − DLQ = 20.10 − 2.38 − 2.54 = 15.18

205. In the usual form of computation, the difference in the height and time of each tide at the two stations is computed, and any tide showing an abnormal difference in height or time is rejected. When a record of the actual high and low waters at a suitable base station is not available, a comparison based on the predicted tides at a suitable station, as given in the tide tables, affords a better determination than a short-term record at the secondary station alone, although clearly not as reliable as a comparison with the actual tides at the base station.

206. *Precision of a determination by comparison.*—The precision of a determination of tidal datums by comparison depends on the water distance between the stations, the freedom of the movement of the tide between them, and the length of the observations. When the secondary station is only a few miles from the base station, on an open and unrestricted waterway, observations extending over a short time

will give the datum with the precision to which the gages can be read. To establish the datum for a project survey of a harbor or waterway where no reliable datum is available, the observations should extend over at least one period of 29 days. Such a comparison should establish the datum within a tenth of a foot if the base station is not too remote. If less accuracy is needed, a comparison for a week may be sufficient.

207. In general a comparison with a suitable base station extending over a year will give a determination of mean sea level within 0.05 foot of the long-term mean at the secondary station, and a comparison extending over 4 years within 0.02 foot. (Special Publication 135, U. S. Coast and Geodetic Survey.) The determination of mean sea level at a station where a long record is not available is always improved by comparing it with a primary station.

FIXATION OF DATUM PLANES

208. It has been seen that even the long-term means of the elevations of the various tidal datums change slightly as the records are extended. Since changes in the datum on which successive surveys are based tends to confusion and error, and makes a comparison between surveys a difficult and laborious process, an accepted elevation of the adopted datum is established with respect to a stable bench mark, or preferably a group of bench marks, as soon as this datum is determined with sufficient precision. This datum is not thereafter changed, unless new conditions make it grossly erroneous.

209. *Accuracy required.*—So far as the usual purposes of navigation and of harbor improvement are concerned, no high degree of precision is required in the determination of a reference datum. The surface of tidal waters is constantly changing in elevation, and may occasionally be a foot or more below any of the datums used in the United States. The squat of a vessel underway, and its pitch in rough water, also render useless any refinements in the indicated depths. Hydrographic charts therefore show the depths of inshore soundings to the nearest foot, and offshore soundings in shoal areas to the nearest quarter fathom (1.5 feet), and to the nearest fathom in deep water. Channel depths are usually laid out to the nearest foot, although ordinarily the sounding from which estimates of dredging are prepared are taken to the nearest tenth of a foot. The fixation of the reference datum within a tenth of a foot or more of its true long-term mean is therefore ordinarily sufficient. The stabilization of the datum is more important than its inherent accuracy.

210. *Datums for dredging contracts.*—In the administration of dredging by contract the definite fixation of the datum plane to cited shore bench marks is essential. If the material removed is measured and

paid for in place, as computed from soundings taken before and after dredging, the systematic error introduced by the variation of even a tenth of a foot between the datums of the two surveys might, in a large contract, result in overpayments or under payments amounting to thousands of dollars. Even when the material is measured in the scows into which it is loaded, the difference in the deductions made for material removed below the plane of tolerance for payment might amount to a considerable sum. The reference bench marks cited in the specifications should be verified before the specifications are issued.

211. Unless the tidal range differs materially in different parts of a harbor or waterway, the datum for harbor improvement is taken as that at one selected tidal station. The datum at all other points is then taken as at the same elevation, this elevation being determined either by lines or levels on shore, or by water levels established by the half tide level corrected by comparison with the base station. When, however, the tidal range, and consequently the elevation of the adopted low water datum below mean sea level, differs materially along the waterway, a succession of reference planes should be used, each applicable to definitely defined sections or areas, and all correlated to a common datum, preferably mean sea level. Thus on successive sections of the East River, N. Y., some 8 datum planes of mean low water are used, varying in elevation from 1.96 to 3.51 feet below mean sea level.

TIDAL OBSERVATIONS

212. *Staff and automatic gages.*—Tidal observations to establish tidal datums, to provide the data for the harmonic analysis of the tide, or to show the varying height of the water with respect to the datum during surveys and dredging operations, are taken on staff or on automatic gages. The *staff gage* is a graduated board, usually set vertically, on which the height of tide is read by an observer. It is ordinarily graduated in feet and tenths, with bold markings so that it can be read at a distance. An *automatic gage* is a device by which the elevation of a float is recorded, on a reduced scale, on a moving paper driven by clockwork. The float is enclosed in a box or pipe, with a restricted entrance near the bottom, to dampen the fluctuations due to wind waves. In cold climates this box is filled with kerosene to prevent freezing. A staff gage is always installed with an automatic gage, the zero of the staff establishing by direct comparison the zero of the record. Two types of automatic gages have been developed by the United States Coast and Geodetic Survey, one a more elaborate instrument for permanent stations at which long-term records are maintained, and the other a portable type for the temporary occupation of a station.

213. *Uses.*—The automatic gage is especially useful for the establishment of tidal datum planes, for securing the data for the harmonic analysis of the tides, and for hydrographic surveying when it is not possible to establish a gage within sight of the area surveyed. The staff gage is usually more convenient for determining the varying elevation of the water during surveys made in the vicinity of the gage, and for regulating the operation of dredges, since the elevations are immediately available and can be read from a distance. For establishing a low water datum by comparison, only the heights of the high and low waters need be taken off the record; although the times of their occurrence may be taken off also as a check, or to determine the lunitidal intervals, if a determination of the latter is desired.

214. *Reference to bench marks.*—A staff gage is easily destroyed and usually lasts for but a short time, unless at least it is built into a permanent structure. Even in the latter case the structure may settle or suffer enough disintegration to displace the zero of the gage. The record of an automatic gage is dependent on its accompanying staff gage. No tide gage serves much useful purpose, therefore, unless its zero is referred to stable shore bench marks, and if a valuable record is desired it should be referred to at least three bench marks well separated from each other. Staff gages for surveys and for the operation of dredges are ordinarily set from bench marks, with their zero at the established datum.

215. *Operation of an automatic gage.*—No clock keeps perfect time, and a clock mechanism driving a relatively heavy recording device cannot be expected to. The registering apparatus of the gage may bend or lag or get out of order, the intake and the well may clog, the float may leak and the wharf or other structure on which the gage is installed may settle. An automatic gage must therefore be tended daily to see that it is functioning properly, and the height of the tide on the staff, with the time at which it is taken, inscribed on the record of the gage. The gage must be inspected by an engineer at intervals, and the zero of the staff gage checked against the reference bench marks. The detailed technique for the installation and operation of automatic tide gages and the tabulation of the record, is given in a Manual of Tide Observation, Special Publication 196, United States Coast and Geodetic Survey. Because of the cost of securing them, reliable records of the tide over any considerable period are available only at a relatively limited number of stations in the harbors of the United States; but these are sufficient to afford a good determination, by comparison, of the datums at any point on the coast line.

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TIDAL CURRENTS

CHAPTER V

RELATION OF CURRENT TO SURFACE SLOPE

216. *General equation for varying flow in a channel.*—The velocity of the current in a tidal channel is continuously increasing or decreasing, and the direction of the flow is periodically reversed. To become applicable to tidal flow, the familiar equations for steady flow must therefore be elaborated to account for the work done in the acceleration and deceleration of the current. The most casual consideration of tidal flow shows, however, that in a channel whose width and depth are small in comparison with the length, the lateral and vertical movements of the water may be neglected, as they are in the equations for steady flow. Similarly, in the derivation of the equations for tidal flow, the velocity at a given instant may be taken as of the same value throughout a cross section of the channel perpendicular to the channel axis.

217. *Units.*—In the ensuing development and application of the equations for tidal flow, the time, t, will be expressed in seconds, unless otherwise stated; lengths, heads, and other dimensions in feet; velocities in feet per second and acceleration in feet per second per second. Conforming to these units, the speeds of the harmonic components (par. 49) are derived in radians (or degrees) per second; but in the application of the formulas, it ordinarily will be more convenient to convert these speeds into degrees per hour.

218. *Derivation of equation of motion.*—Taking the X axis of coordinates as a horizontal line in the direction of the axis of the channel, and the Y axis as vertical, let:

v be the velocity of the current at the time t, and at a cross section of the channel distant x from the origin of coordinates; v is taken as positive when the direction of flow is in the positive direction of x, and negative when the flow is in the opposite direction.

$\partial v/\partial t$, the acceleration of the velocity at a given cross section with respect to time.

$\partial v/\partial x$, the rate at which the velocity is increasing (algebraically) at a given instant with the distance of the cross section from the origin.

dx, the distance, along the direction of the X axis, traveled by a particle of water during the elementary time interval dt.

$(\partial y/\partial x)dx$, the (algebraic) increase in the elevation of the water surface in the distance dx; this increase being positive if the slope of the water surface is upward, and negative if downward in the direction x positive.

X, the area of the cross section of water prism of the channel, at the point x, and at the time t.

Q, the discharge through the cross section.

w, the weight of 1 cubic foot of water.

g, the acceleration due to gravity.

m, the mass of the water discharged through the cross section during the time interval dt.

r, the hydraulic radius of the channel at the section under consideration.

C, the Chezy coefficient applicable to this section.

Then: $v=dx/dt$

$$Q=Xv=Xdx/dt.$$

The volume of the discharge, during the time dt is $Qdt=Xdx$, and its mass, m, is:

$$m=wQdt/g=wXdx/g.$$

The mass of the water in an elementary section of the channel of length dx is also:

$$wXdx/g=m.$$

219. During the time interval dt, work is done in an elementary section of the channel of length dx:

(*a*) In raising the mass of the discharge the distance $(\partial y/\partial x)dx$ in its passage through the channel.

The work so done is:

$$mg(\partial y/\partial x)dx.$$

(*b*) In increasing the kinetic energy of this mass because of the increase $(\partial v/\partial x)dx$, in its velocity in the distance dx. The kinetic energy of the moving mass is $mv^2/2$. The force required to increase this energy is:

$$\partial(mv^2/2)/\partial x = mv\partial v/\partial x$$

and the work done by this force over the distance dx is:

$$mv(\partial v/\partial x)dx.$$

(*c*) In the acceleration of the mass of water in the section, with respect to time. The work so done is:

$$m(\partial v/\partial t)dx.$$

(*d*) In overcoming frictional resistance in the section. The frictional resistance in a channel is due to the turbulence which the flow produces and is dependent upon the velocity of the current. The turbulence created at any instant by the slowly varying velocity in a tidal channel cannot differ sensibly from that which would be produced by the same constant velocity. The work done in overcoming frictional resistance in the section of length dx may then be taken as that developed from the usually accepted formula for steady flow. This work is $mg(v^2/C^2r)dx$. To become applicable to the reversing flow in tidal channels, the algebraic sign of this expression must be considered, since the work is positive when the flow is in the positive direction of x, and negative when the flow is in the opposite direction. Since v^2 does not change its sign in passing through zero, and the other quantities are not directional, this item of work will be written:

$$\pm mg(v^2/C^2r)dx.$$

The positive sign is to be applied when v is positive, and the negative sign when v is negative.

220. Each of the items of work developed in the preceding paragraph may be either positive or negative. The work done in raising the mass of the discharge (item *a*) is positive if $\partial y/\partial x$ is positive, and negative if negative. Item (*b*) is positive if the kinetic energy is increasing in the positive direction of x, and negative if decreasing, while item (*c*) changes its sign with $\partial v/\partial t$, and item (*d*) with v.

Since no external work is done by the flow in the channel, the sum of all of the items must be zero, giving:

$$mg(\partial y/\partial x)dx + mv(\partial v/\partial x)dx + m(\partial v/\partial t)dx \pm mg(v^2/C^2r)dx = 0$$

Dividing by $mgdx$, this equation reduces to:

$$\partial y/\partial x + (v/g)\partial v/\partial x + (1/g)\partial v/\partial t \pm v^2/C^2r = 0. \qquad (112)$$

Thus is the basic equation of motion in a tidal channel.

221. *Discussion.*—The first term, $\partial y/\partial x$, in equation (112) is the slope of the water surface in the channel at the given cross section and

at the given time. The second term, $(v/g)\partial v/\partial x$, is the rate of change of $v^2/2g$ and may therefore be regarded as the component slope due to the velocity head. The third term represents the effect of the acceleration or deceleration of the current, and the last term is due to the frictional resistance.

The velocity, v, at a given instant, varies in fact from point to point in a cross section of a channel carrying tidal flow, as it does in a cross section of a channel when the flow is steady. In both cases, v is taken as the mean velocity at the section.

In the derivation of equation (112) the flow is regarded as continuously turbulent, even during the short interval in which the velocity becomes very small in passing through zero, as the current reverses. It is evident, however, that a change in the character of the flow during so brief a period may be disregarded, even if such change in fact occurs.

222. *Application of general equation to steady uniform flow.*—When the flow is steady and uniform, the velocity throughout the channel remains constant, and $\partial v/\partial x$ and $\partial v/\partial t$ are zero. Taking the velocity as in the positive direction, equation (112) reduces to

$$\partial y/\partial x + v^2/C^2r = 0. \tag{113}$$

Designating the slope of the water surface as s, and observing that when the flow is steady the slope is downward, so that $\partial y/\partial x = -s$, equation (113) becomes:

$$-s + v^2/C^2r = 0.$$

Whence

$$v = C\sqrt{rs}. \tag{114}$$

Equation (114) is the generally accepted basic formula for steady flow, in which the Chezy coefficient, C, may be determined from the Kutter, Bazin, Manning, or other formulas.

223. *Selection of Chezy coefficient for tidal flow.*—It is apparent from the preceding discussion that the value of C to be used in equation (112) when the flow is tidal should be that applicable to the channel were the flow steady. While the value of C determined from the Kutter formula varies somewhat with the slope in the channel, and this slope fluctuates between limits when the flow is tidal, this variation in C is so small with the slopes usually found in tidal channels that either the maximum or the numerical mean or median slope during the tidal cycle may be used in the application of the formula without affecting the value of C to a greater degree than that inherent in the uncertainty in the selection of the proper coefficient of roughness.

224. *Expression for the friction term when the velocity has a harmonic fluctuation.*—It has been seen that the friction term in equation (112) changes its sign in passing through zero, while the expression for the friction term, v^2/C^2r, does not change its sign. A mathematically con-

tinuous expression for the friction term may be derived when the velocity has the simple harmonic variation:

$$v = B \sin (at + \beta), \tag{115}$$

in which B is the maximum numerical value of v during the tidal cycle. Designating the friction term as F, then:

$$F = \pm (B^2/C^2 r) \sin^2 (at + \beta). \tag{116}$$

The positive sign is to be applied when $(at+\beta)$ has values between 0 and π, 2π and 3π, 4π and 5π, etc.; and the negative sign when $(at+\beta)$ has values between π and 2π, 3π and 4π, etc.

The graph of such a function is shown by the solid line in figure 36.

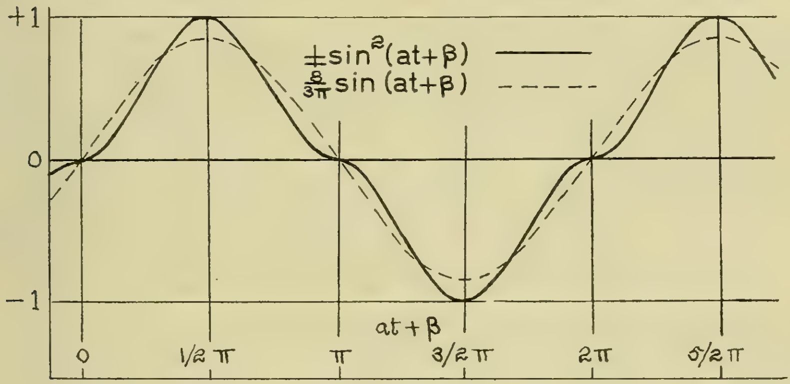

FIGURE 36.—Graph of friction term of an harmonic current.

225. Placing, for convenience, $at+\beta=x$, the function $\pm\sin^2 x$ is by definition such that $\sin^2 (-x) = -\sin^2 x$. By Fourier's theorem it should therefore be expressed by the series:

$$A_1 \sin x + A_2 \sin 2x + A_3 \sin 3x + \ldots A_n \sin nx \ldots \tag{117}$$

In which the coefficients, for values of x between 0 and π, are:

$$A_1 = (2/\pi)\int_0^\pi \sin^2 x \sin x \, dx, \; A_2 = (2/\pi)\int_0^\pi \sin^2 x \sin 2x \, dx,$$

$$\ldots A_n = (2/\pi)\int_0^\pi \sin^2 x \sin nx \, dx.$$

And, for values of x between π and 2π, the coefficients are:

$$A_1 = (2/\pi)\int_\pi^{2\pi} \sin^2 x \sin x \, dx, \; A_2 = (2/\pi)\int_\pi^{2\pi} \sin^2 x \sin 2x \, dx,$$

$$\ldots A_n = (2/\pi)\int_\pi^{2\pi} \sin^2 x \sin nx \, dx$$

In these expressions, n is any integer. Since:

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$, and $\cos 2x \sin nx = \frac{1}{2}\sin(2+n)x - \frac{1}{2}\sin(2-n)x$,

$$A_n = (2/\pi)\int_0^\pi \sin^2 x \sin nx\,dx$$

$$= (1/\pi)\int_0^\pi \sin nx\,dx - (1/\pi)\int_0^\pi \cos 2x \sin nx\,dx$$

$$= (1/\pi)\int_0^\pi \sin nx\,dx - (1/2\pi)\int_0^\pi \sin(2+n)x\,dx$$

$$+ (1/2\pi)\int_0^\pi \sin(2-n)x\,dx$$

$$= -\frac{\cos nx}{n\pi}\Big]_0^\pi + \frac{\cos(n+2)x}{2(n+2)\pi}\Big]_0^\pi + \frac{\cos(n-2)x}{2(n-2)\pi}\Big]_0^\pi \tag{118}$$

The values of $\cos nx$, $\cos(n+2)x$ and $\cos(n-2)x$ are $+1$ when $x=0$. If n is odd their value is -1 when $x=\pi$; but if n is even their value is $+1$ when $x=\pi$. Therefore, for values of x between 0 and π, the value of A_n is, when n is odd:

$$A_n = \frac{2}{n\pi} - \frac{1}{(n+2)\pi} - \frac{1}{(n-2)\pi} = -\frac{8}{n(n^2-4)\pi}, \tag{119}$$

but when n is even, $A_n=0$.
Substituting successive odd values of n:

$$A_1 = 8/(3\pi),\ A_3 = -8/(15\pi),\ A_5 = -8/(105\pi),\ A_7 = -8/(315\pi), \text{ etc.}$$

For values of x between 0 and π, therefore:

$$\sin^2 x = (8/3\pi)(\sin x - 1/5 \sin 3x - 1/35 \sin 5x - 1/105 \sin 7x \ldots) \tag{120}$$

Similarly, for values of x between π and 2π:

$$A_n = -\frac{\cos nx}{n\pi}\Big]_\pi^{2\pi} + \frac{\cos(n+2)x}{2(n+2)\pi}\Big]_\pi^{2\pi} + \frac{\cos(n-2)x}{2(n-2)\pi}\Big]_\pi^{2\pi} \tag{121}$$

Since, when n is odd, the functions $\cos nx$, $\cos(n+2)x$ and $\cos(n-2)x$ have a value of -1 when $x=\pi$ and of $+1$ when $x=2\pi$, the value of A_n becomes, between these limits.

$$A_n = +\frac{8}{n(n^2-4)\pi} \tag{122}$$

But when n is even the values of these cosine functions is $+1$ both when $x=\pi$ and when $x=2\pi$, and the coefficient is zero.

For values of x between π and 2π, therefore;

$$\sin^2 x = -(8/3\pi)(\sin x - 1/5 \sin 3x - 1/35 \sin 5x - 1/105 \sin 7/x \ldots) \quad (123)$$

It similarly may be shown that for values of x between 2π and 3π the expression for $\sin^2 x$ is given by equation (120); for values of x between 3π and 4π, by equation (123) and so on.

226. Therefore, when $v = B \sin (at + \beta)$ the value of F is represented by the continuous function:

$$F = (8/3\pi)(B^2/C^2r)[\sin (at+\beta) - 1/5 \sin 3(at+\beta) - 1/35 \sin 5 (at+\beta) \ldots] \quad (124)$$

The friction term is then the resultant of a principal component,

$$F_1 = (8/3\pi)(B^2/C^2r) \sin (at+\beta) = (8/3\pi) Bv/C^2r \quad (125)$$

with the speed of the velocity, and minor components whose speeds are 3, 5, 7, etc., times the speed of the principal component. The correspondence between the principal component and the complete value of F is shown in figure 36.

The derivation of a mathematically continuous expression for F when the velocity is the resultant of two or more harmonic components would be difficult, if not impossible.

SURFACE, VELOCITY, ACCELERATION, AND FRICTION HEADS

227. It is the generally accepted practice to apply the formulas for steady flow to sections or reaches of a channel of considerable length, even though the velocity is not entirely uniform throughout such reaches because of a variation of successive cross sections of the water prism in the channel. For the computation of the friction term, the velocity throughout the reach is taken as the average velocity, as determined usually by the discharge through the average cross section. The error introduced by the assumption, as well as the error introduced by considering the velocity at any point in the channel as the mean velocity in the cross section, is generally small in comparison with the uncertainty in the selection of the proper coefficient of roughness to derive the value of C. The equation for varying flow may similarly be applied to sections of channel of considerable length, so long as the velocity and the slope at any instant are tolerably constant throughout the section. In deep channels these conditions are fulfilled in sections several miles in length. Denoting the length of such a section by l, equation (112) establishes the relationship:

$$l\partial y/\partial x + l(v/g)\partial v/\partial x + (l/g)\partial v/\partial t \pm lv^2/C^2r = 0. \quad (126)$$

228. The end of the section from which distances in the section extend in the positive direction may be designated the *initial end.*

The elevation of the water surface at the initial end of the section at the time t will be designated y_0, at the other end at the same instant, y_1; the velocity at the initial end, v_0, and the velocity at the other end, v_1.

229. In the first term of equation (126) the slope, $\partial y/\partial x$, may be considered the average slope through the section. This term is then the difference, y_1-y_0, between the elevations of the water surface at the ends of the section: it will be called the *surface* head and designated h_s. Then

$$h_s=l\partial y/\partial x=y_1-y_0 \tag{127}$$

It may be noted that the surface head is positive when the water is sloping *upward* in the positive direction along the channel, and negative when sloping downward.

230. The second term of equation (126), $(v/g)\partial v/\partial x$ is the change in $v^2/2g$ between the ends of the section, and is therefore the *velocity* head, h_v, giving:

$$h_v=l(v/g)\partial v/\partial x=v_1^2/2g-v_0^2/2g \tag{128}$$

It may be noted that the velocity head is positive when the velocity is increasing numerically in the positive direction, and negative when the velocity is numerically decreasing in that direction.

231. The third term of the equation may be called the *acceleration* head, and designated h_a, giving

$$h_a=(l/g)\partial v/\partial t \tag{129}$$

In which $\partial v/\partial t$ is the acceleration of the average velocity through the section.

The acceleration head is positive when the average velocity through the section is increasing *algebraically* with respect to time.

232. The fourth term is the *friction head*, h_f, giving:

$$h_f=\pm lv^2/C^2r \tag{130}$$

in which v may be taken as the average velocity through the section at the given instant, C is the Chezy coefficient applicable to the section, and r the hydraulic radius. The friction head has the same sign as the velocity.

233. Substituting the symbols for the various heads in equation (126):

$$h_s+h_v+h_a+h_f=0 \tag{131}$$

In this equation, h_s, h_v, and h_f have the same significance as in steady flow, except that h_s and h_f, as well as h_v may be negative. The acceleration head is the additional term which must be included

in tidal flow. It is readily evaluated when the rate of change in the velocity is known. Thus if the velocity is algebraically decreasing at a given time at the rate of 0.003 feet per second per second in a section of channel 10,000 feet in length, the acceleration head is $-10{,}000 \times 0.003/32.16 = -0.93$ feet.

234. *Entrance and recovery heads.*—In steady flow, the head due to the increase in velocity at the entrance to a contracted section of a channel is termed the entrance head. Its commonly accepted value is:

$$h = m(v_1^2 - v_0^2)/2g \qquad (132)$$

In this equation v_0 is the velocity in the approach to the contraction, v_1 the velocity in the contracted section, and m a coefficient to account for the increased turbulence at the contraction. The entrance head is then the change in $v^2/2g$ at the entrance, times a suitable coefficient. If the velocity in the contracted section is not large, m is often taken as unity.

At the outlet of the contracted section, the decrease in the kinetic energy of the flowing water gives rise to a recovery of head whose value is given by the same formula by taking v_0 as the velocity in the contracted section, and v_1 the velocity in the expanded channel. The recovery head is then also the change, in the positive direction, of $v^2/2g$, times a suitable coefficient. Since, however, the recovery of energy is never complete, the value of m for the recovery head is always less than unity, and frequently is taken as 0.5.

235. In reversing tidal flow each end of a contracted section of channel is alternately the entrance and the outlet. At the initial end the flow is into the contracted section when the velocity is positive, and out when the velocity is negative; but since the numerical value of the velocity remains the greater in the contracted section, the change in $v^2/2g$ at the entrance is positive, whichever the direction of the flow. At the other end the flow is out of the contracted section when the velocity is positive, and into the section when negative; but the change (in the positive direction) of $v^2/2g$ is always negative. At both ends, therefore, the change (in the positive direction) of $v^2/2g$ represents an entrance head when it has the same sign as the velocity, and a recovery head, to which a reducing factor should be applied, when it is of opposite sign to the velocity. On an expanded section of the channel, the contrary condition obviously exists.

236. *Contraction heads.*—At a sudden local contraction in a section of channel that otherwise may be taken as uniform, such as at a bridge, the increased turbulence resulting from the increase and decrease of the current produces a net head, of the same nature as an increase in the friction head. Such a contraction head may be introduced in computations of tidal flow by determining, from the applicable for-

mulas developed for steady flow, its numerical value at the given velocity in the section, and giving it the sign of the velocity.

CURRENTS PRODUCED BY A SIMPLE HARMONIC FLUCTUATION OF THE SURFACE HEAD AND SLOPE IN A SHORT SECTION OF A CHANNEL

237. The surface head in a section of a tidal channel is the difference between the elevation of the tides at the ends of the section. These tides are, as has been seen, the resultant of a number of harmonic components, of various amplitudes, speeds, and phases, occasionally modified by meteorological disturbances. Quite obviously, the surface head likewise is the resultant of harmonic components of the same speeds; but it does not follow that the amplitudes of the components of the head are proportional to the amplitudes of the tides. On the contrary, the amplitudes of the diurnal components of the head may be, and frequently are, proportionally less than the amplitudes of the diurnal components of tides of the mixed type; and overtides of the head often are relatively more important than those of the tides themselves. Generally, the head in a short section of a tidal channel during a single tidal cycle does not depart widely from a simple harmonic fluctuation with the speed, m_2, of the principal lunar component of the tides. The currents derived from such a simple harmonic fluctuation of the head often afford a sufficient indication of the strength and timing of the actual currents; and in any case provide a basis for a determination of the currents resulting from a head which has a given variation from the simple harmonic fluctuation assumed.

238. *Relation of surface head to a simple harmonic fluctuation of the tides at the ends of a short section of channel.*—If the tides at the ends of the section are taken to have simple harmonic fluctuations of the same speed, the head likewise has a simple harmonic fluctuation of the same speed. Let O and A be two stations on a tidal channel, at such a limited distance, l, apart that at any instant the variation in the velocity and slope between the stations is immaterial. Let:

$$y_0 = A_0 \cos (at + \alpha_0) \tag{133}$$

be the elevation of the water surface at O, taken as the initial station, and let:

$$y_1 = A_1 \cos (at + \alpha_1) \tag{134}$$

be the elevation of the water surface at A.

The surface head in the section is then:

$$\begin{aligned} h_s=y_1-y_0&=A_1 \cos (at+\alpha_1)-A_0 \cos (at+\alpha_0) \\ &=A_1 \cos (at+\alpha_1)+A_0 \cos (at+\alpha_0+180°) \end{aligned} \tag{135}$$

Since two components of the same speed unite into a single component of that speed, equation (135) may be written:

$$h_s=H \cos (at+H^0) \tag{136}$$

In which H is the amplitude, and H^0 the initial phase, of the fluctuation of the surface head during the tidal cycle.

239. *Computation of H and H^0.*—The amplitude and phase of the surface head in the section readily may be determined from the amplitudes and phases of the tides at the ends of the section, through the relation established in equations (135) and (136):

$$H \cos (at+H^0)=A_1 \cos (at+\alpha_1)-A_0 \cos (at+\alpha_0) \tag{137}$$

Equation (137) is identically true for all values of t. By placing $at=0$, the equation of condition is derived:

$$H \cos H^0=A_1 \cos \alpha_1-A_0 \cos \alpha_0 \tag{138}$$

and by placing $at=-90°$

$$H \sin H^0=A_1 \sin \alpha_1-A_0 \sin \alpha_0 \tag{139}$$

The values of H^0 and H may then be determined from the equations:

$$\tan H^0=H \sin H^0/H \cos H^0 \tag{140}$$

$$H=H \sin H^0/\sin H^0=H \cos H^0/\cos H^0 \tag{141}$$

The amplitude, H, is directionless. The quadrant in which H^0 lies is determined by the algebraic signs of $H \sin H^0$ and $H \cos H^0$.

240. *Example.*—The curve showing the average height of the tide at station 180+30 on the Cape Cod Canal, after the time of a lunar transit, prepared from observations during the period September 28 to October 6, 1932, is represented by the equation:

$$y=3.74 \cos (\mathrm{m}_2t+58°34')$$

and at station 225 by:

$$y=3.18 \cos (\mathrm{m}_2t+61°10')$$

Taking station 180+30 as the initial station:

$$A_1 \cos \alpha_1 = 3.18 \cos 61°10' = 1.534 \qquad A_1 \sin \alpha_1 = 3.18 \sin 61°10' = 2.786$$

$$A_0 \cos \alpha_0 = 3.74 \cos 58°34' = 1.950 \qquad A_0 \sin \alpha_0 = 3.74 \sin 58°34' = 3.191$$

$$H \cos H^0 = -.416 \qquad H \sin H^0 = -.405$$

$$\tan H^0 = 0.405/0.416 = 0.9736$$

The corresponding angle, from a table of natural tangents, is 44°14′. Since the sine and cosine are both negative, H^0 lies in the third quadrant, and is:

$$H^0 = 180° + 44°14' = 224°14'$$

and

$$H = 0.416/\cos 44°14' = 0.58$$

The equation of the surface head in the section between the two stations is therefore:

$$h_s = 0.58 \cos (m_2 t + 224°14')$$

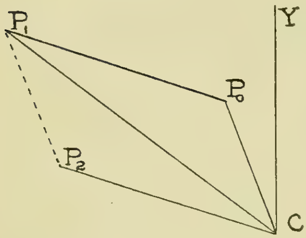

FIGURE 37.—Relation of head to tides.

241. *Generating radius of head.*—The relation between the generating radii of the curves representing the tidal heights at the two ends of the channel and that of the head in the channel is shown in figure 37, in which $CP_0 = A_0$ is the generating radius of the tide at the initial end, $CP_1 = A_1$ that at the other end of the channel, $CP_2 = P_0P_1$ is the generating radius of the curve showing the surface head.

242. *Equation of primary current.*—As will later be made apparent, the currents produced by a simple harmonic fluctuation of the surface head in a short section of the channel depart somewhat from a simple harmonic fluctuation. These distortions of the current are due to the form of the velocity head term, $(v/g)\,\partial v/\partial x$, in the general equation of motion, to the minor components of the friction term produced by a harmonically varying current (par. 226) and to the variation in the hydraulic radius and Chezy coefficient with the rise and fall of the tide. Under usual conditions of tidal flow the velocity head in a short section of a channel is relatively so small that the velocity head term may be omitted. The other disturbing elements may be treated

as corrections to a harmonically varying *primary* current, represented by equation (115)

$$v = B \sin (at + \beta)$$

243. Dropping the velocity head term, $(v/g) \partial v/\partial x$, from equation (112), and substituting for the friction term its principal component for an harmonically varying current (equation 125), the differential equation of the primary current is:

$$\partial y/\partial x + (1/g)\partial v/\partial t + (8/3\pi)Bv/C^2 r = 0 \tag{142}$$

In this equation C and r are the values of the Chezy coefficient and hydraulic radius at mean tide.

In equation (142):

$$\partial y/\partial x = h_s/l = (H/l) \cos (at + H^0) = S \cos (at + H^0), \tag{143}$$

in which $S = H/l$ is the numerical value of the maximum slope in the channel during the tidal cycle.

From equation (115):

$$\partial v/\partial t = aB \cos (at + \beta) \tag{144}$$

Equation (142) then becomes:

$$S \cos (at + H^0) + (aB/g) \cos (at + \beta) + (8/3\pi)(B^2/C^2 r) \sin (at + \beta) = 0 \tag{145}$$

In which a is expressed in radians per second.

244. *Solution of equation.*—By placing $at = 0$ and $at = -\pi/2$ in equation (145), two equations of condition are established from which expressions for B and β may be derived. When $at = 0$, equation (145) reduces to:

$$S \cos H^0 + (aB/g) \cos \beta + (8/3\pi)(B^2/C^2 r) \sin \beta = 0 \tag{146}$$

and when $at = -\pi/2$, to:

$$S \sin H^0 + (aB/g) \sin \beta - (8/3\pi)(B^2/C^2 r) \cos \beta = 0 \tag{147}$$

Multiplying equation (146) by $\cos \beta$ and equation (147) by $\sin \beta$ and adding:

$$S(\cos H^0 \cos \beta + \sin H^0 \sin \beta) + (aB/g)(\cos^2 \beta + \sin^2 \beta) = 0$$

or:

$$S \cos (H^0 - \beta) + Ba/g = 0 \tag{148}$$

Multiplying equation (146) by $\sin \beta$ and equation (147) by $\cos \beta$ and subtracting:

$$S(\cos H^0 \sin \beta - \sin H^0 \cos \beta) + (8/3\pi)(B^2/C^2r)(\sin^2 \beta + \cos^2 \beta) = 0$$

or:

$$-S \sin (H^0 - \beta) + (8/3\pi)B^2/C^2r = 0 \tag{149}$$

It is convenient to place:

$$H^0 - \beta = \phi + \pi/2 \tag{150}$$

So that equations (148) and (149) become:

$$S \sin \phi = aB/g \tag{151}$$

$$S \cos \phi = (8/3\pi)B^2/C^2r \tag{152}$$

Whence

$$B/C^2r = (3\pi/8)(a/g) \cot \phi \tag{153}$$

Eliminating B from equations (151) and (153):

$$\sin \phi \tan \phi = (3\pi/8)(a/g)^2 C^2 r/S \tag{154}$$

And from equation (152):

$$B = \sqrt{3\pi/8}\, C \sqrt{rS} \sqrt{\cos \phi} \tag{155}$$

Or, from equation (151):

$$B = (g/a)\, S \sin \phi \tag{156}$$

It may be seen from equations (151) and (152) that both $\sin \phi$ and $\cos \phi$ are intrinsically positive. ϕ is therefore an angle between 0 and 90°.

245. *Computation of ϕ and B.*—Equation (154) may be written:

$$(g/a)\sqrt{\sin \phi \tan \phi} = \sqrt{3\pi/8} \cdot C\sqrt{rS}/S \tag{157}$$

Placing for convenience,

$$P = \sqrt{3\pi/8}\, C \sqrt{rS} \tag{158}$$

$$= 1.0854\, C\sqrt{rS}$$

This equation reduces to:

$$(g/a)\sqrt{\sin \phi \tan \phi} = P/S \tag{159}$$

The numerical values of P and P/S are readily computed from the amplitudes, S, of the slope in the section during the tidal cycle, and the Chezy coefficient C and hydraulic radius, r, at mean tide.

TABLE IX

ϕ	P/S	log P/S	$\sqrt{\cos\phi}$	log $\sqrt{\cos\phi}$	ϕ	P/S	log P/S	$\sqrt{\cos\phi}$	log $\sqrt{\cos\phi}$
0	0		1	0					
1°	3,995	3.60147	1.000	9.99997	46°	197,500	5.29563	.833	9.92088
2°	7,990	3.90253	1.000	9.99986	47°	202,700	5.30681	.826	9.91689
3°	11,990	4.07868	.999	9.99970	48°	207,900	5.31790	.818	9.91275
4°	15,980	4.20369	.999	9.99947	49°	213,300	5.32889	.810	9.90847
5°	19,980	4.30070	.998	9.99917	50°	218,700	5.33980	.802	9.90403
6°	23,990	4.38002	.997	9.99880	51°	224,200	5.35065	.793	9.89943
7°	28,000	4.44710	.996	9.99837	52°	229,900	5.36144	.785	9.89467
8°	32,010	4.50526	.995	9.99788	53°	235,600	5.37220	.776	9.88973
9°	36,030	4.55660	.994	9.99731	54°	241,500	5, 38292	.767	9.88461
10°	40,050	4.60257	.992	9.99667	55°	247,500	5.39365	.757	9.87929
11°	44,080	4.64420	.991	9.99597	56°	253,700	5.40437	.748	9.87378
12°	48,110	4.68226	.989	9.99520	57°	260,100	5.41512	.738	9.86805
13°	52,160	4.71731	.987	9.99436	58°	266,600	5.42589	.728	9.86210
14°	56,210	4.74980	.985	9.99345	59°	273,400	5.43672	.718	9.85592
15°	60,270	4.78010	.983	9.99247	60°	280,300	5.44762	.707	9.84948
16°	64,340	4,80850	.980	9.99142	61°	287,500	5.45861	.696	9.84278
17°	68,430	4.83522	.978	9.99030	62°	294,900	5.46971	.685	9.83580
18°	72,520	4.86046	.975	9.98910	63°	302,600	5.48094	.674	9.82852
19°	76,630	4.88438	.972	9.98783	64°	310,700	5 49232	.662	9.82092
20°	80,750	4.90714	.969	9.98649	65°	319,100	5.50388	.650	9.81297
21°	84,890	4.92883	.966	9.98507	66°	327,800	5.51565	.638	9.80466
22°	89,040	4.94957	.963	9.98358	67°	337,000	5.52767	.625	9.79594
23°	93,210	4.96944	.959	9.98201	68°	346,700	5.53996	.612	9.78679
24°	97,390	4.98853	.955	9.98036	69°	356,900	5.55257	.599	9.77716
25°	101,600	5, 00689	.952	9.97864	70°	367,700	5.56554	.585	9.76702
26°	105,800	5.02459	.948	9.97683	71°	379,300	5.57893	.571	9.75632
27°	110,100	5, 04168	.944	9.97494	72°	391,600	5.59279	.556	9.74499
28°	114,300	5, 05822	.940	9.97297	73°	404,800	5.60721	.541	9.73297
29°	118,700	5.07424	.935	9.97091	74°	419,000	5.62225	.525	9.72017
30°	123,000	5, 08978	.931	9.96876	75°	434,500	5.63803	.509	9.70650
31°	127,300	5, 10489	.926	9.96653	76°	451,500	5.65465	.492	9.69184
32°	131,700	5.11958	.921	9.96421	77°	470,200	5.67226	.474	9.67604
33°	136,100	5.13389	.916	9.96179	78°	491,000	5.69104	.456	9.65894
34°	140,500	5, 14785	.911	9.95929	79°	514,300	5.71123	.437	9.64030
35°	145,000	5, 16149	.905	9.95668	80°	540,900	5.73309	.417	9.61983
36°	149,600	5.17482	.899	9.95398	81°	571,500	5.75703	.395	9.59716
37°	154,100	5.18787	.894	9.95117	82°	607,500	5.78356	.373	9.57178
38°	158,700	5.20065	.888	9.94826	83°	650,700	5.81338	.349	9.54295
39°	163,400	5.21320	.882	9.94525	84°	704,000	5.84758	.323	9.50962
40°	168,100	5.22552	.875	9.94212	85°	772,300	5.88778	.295	9.47015
41°	172,800	5.23763	.869	9.93889	86°	864,400	5.93673	.264	9.42179
42°	177,600	5.24955	.862	9.93554	87°	999,000	5.99958	.229	9.35940
43°	182,500	5.26130	.855	9.93206	88°	1,221,300	6.08790	.187	9.27141
44°	187,500	5, 27288	.848	9.92847	89°	1,732,200	6.23859	.132	9 12093
45°	192,500	5.28432	.841	9.92174					

$$P=1.0854C\sqrt{rS} \qquad \log 1.0854=0.03559$$

246. In table IX the values of $P/S=(g/a)\sqrt{\sin\phi\tan\phi}$ with their corresponding logarithms, are tabulated for each degree of ϕ from 0 to 89°, when a is the speed, m_2, of the principal lunar semidiurnal component, in radians per second. From table II, paragraph 75, the value of m_2, in degrees per hour, is 28° .9841. Its value in radians per second is then (28.9841/3600) (π/180)=0.000,140,52. This

value is used in the preparation of the table. The value of g is taken as 32.16. By entering table IX with the computed value of P/S or of its logarithm, the value of ϕ for a simple tidal fluctuation with a speed of m_2, is readily determined by interpolation.

The value of B is then, from equation (155):

$$B=P\sqrt{\cos \phi} \tag{160}$$

247. If the tidal fluctuation has a speed, a, differing from that of the principal lunar semidiurnal component, m_2, on which table IX is constructed, equation (159) may be written:

$$(g/m_2)\sqrt{\sin \phi \tan \phi}=(a/m_2)P/S \qquad (161$$

To determine the value of ϕ, table IX is then entered with the computed value of $(a/m_2)P/S$. It may be noted that the speeds, a and m_2, may be expressed in any common units.

248. *Value of β.*—From equation (150):

$$\beta=H^0-\phi-\pi/2$$

Or, when angles are expressed in degrees:

$$\beta=H^0-\phi-90° \qquad (162$$

The equation of the primary current is then:

$$v=B \sin (at+H^0-\phi-90°) \tag{163}$$

The values of B and ϕ are determined as shown in paragraph 246; and H^0, the initial phase of the head, is determined as shown in paragraph 239.

249. *Examples.*—The surface head between stations 180+30 and 225 in the Cape Cod Canal, at the time t after a lunar transit, was found in paragraph 240 to be:

$$h_s=0.58 \cos (m_2t+224°14')$$

The length of the section is 4,470 feet, giving: $S=0.58/4470=0.000{,}130$. The hydraulic radius, at mean tide, at the time of the observations, is given as 22.7 feet. As the bed is exceptionally rough, an appropriate value of Kutter's "n" is 0.030. Taking the mean slope as 0.0001, the corresponding value of C is 90. From these data:

$$P=1.0854C\sqrt{rS}=5.31, \qquad P/S=40{,}820.$$

From table IX the corresponding value of ϕ is $10°12'$, and from equation (160):

$$B=5.31\sqrt{\cos 10°12'}=5.27 \text{ feet per second.}$$

From equation (163), the primary current, at the time t after a lunar transit is then:

$$v=5.27 \sin (m_2t+124°02')$$

250. The head and the primary current are plotted in figure 38. As station 180+30 has been taken as the initial station, and as the

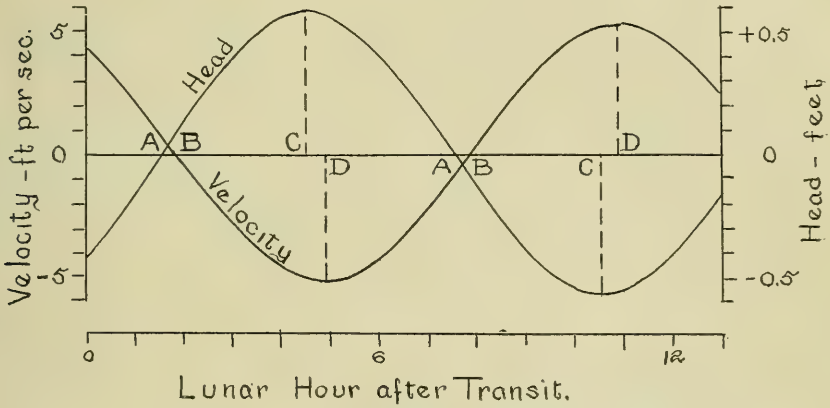

FIGURE 38.—Primary current and head in section of Cape Cod Canal.

stationing is from east to west, westerly currents are positive, and easterly currents are negative.

251. As a second example, the value of S in a section of the Delaware River near the mouth is 0.0000146, the hydraulic radius in the section, at midtide, being 19.3 feet. The value of C corresponding to a coefficient of roughness of 0.025 is 120. Then:

$$P=2.18 \qquad P/S=149{,}300$$

$$\phi=36° \qquad B=2.18\sqrt{\cos 36°}=1.95 \text{ feet per second.}$$

252. The relation between the head in a section 5,000 feet in length and the velocity is shown in figure 39, the origin of time being taken

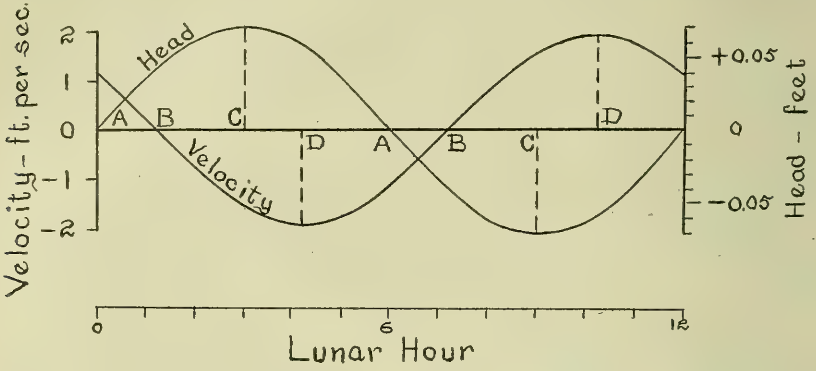

FIGURE 39.—Primary current and head in section of Delaware River near entrance.

at a moment when the head turns from negative to positive.

253. For a third example, a channel may be taken with a hydraulic radius of 100 feet, and a maximum slope, S, of 0.000,01 during the tidal cycle. An appropriate value of C is 150.

Then:

$$P=5.138 \qquad P/S=513,800$$

$$\phi=79° \qquad B=5.135\sqrt{\cos 79°}=2.24 \text{ feet per second.}$$

The relation between the head in a 5,000-foot section of such a channel and the velocity is shown in figure 40, the origin of time being as in the preceding example.

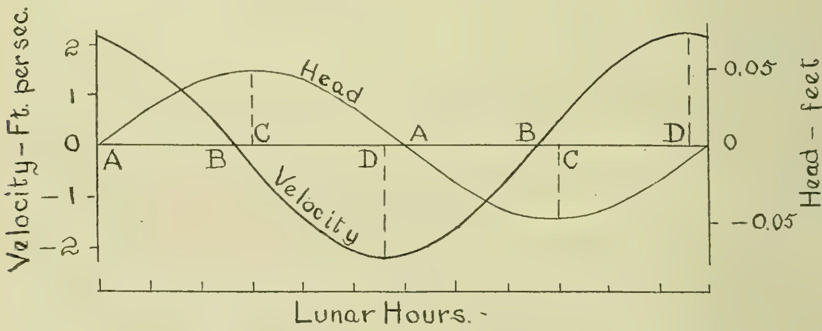

FIGURE 40.—Primary current and head in deep channel.

254. *Lag of primary tidal current.*—It may be observed that positive directions have been so assigned to heights and velocities that the water is running down hill when the head is positive and the velocity

is negative, or vice versa; and is running uphill when both the head and the velocity have the same algebraic sign. While this convention may appear unnatural, it removes the confusion that would result were vertical distances taken as positive in a downward direction. In each of the diagrams illustrating the preceding examples, the head reaches a maximum at the time marked C, and the primary current reaches its strength, in the opposite direction, at a subsequent time marked D. The strength of the primary current in a tidal channel therefore lags behind the maximum surface head and slope by the time interval CD. The turn of the primary current lags behind the turn of the surface head and slope by the equal interval AB.

Designating the time C as t_0, and the time D as t_1, then from the equation of the surface head (equation 136):

$$h_s = H \cos (at + H^0)$$

it is evident that

$$at_0 + H^0 = 0.$$

From the equation of the primary current (equation 163):

$$v = B \sin (at + H^0 - \phi - 90°).$$

and

$$at_1 + H^0 - \phi - 90° = -90°.$$

Whence

$$at_1 - at_0 = \phi.$$

The intervals CD and AB are then equal to ϕ/a; and the angle ϕ may be designated the *angular lag* of the primary current.

255. *Characteristics of tidal flow.*—In each of the preceding examples the water flows downhill during the intervals indicated as BA on the diagrams. At the moments marked A the water surface is level, but the momentum of the moving water continues to carry it in the direction of its motion. During the intervals from A to B the water flows uphill until the momentum is checked. At the instants marked C, when the current reaches its maximum velocity in either direction, the acceleration is zero, and the velocity is determined by the slope and frictional resistance only, and is the same as though the flow were steady; but as the velocity lags behind the head, the maximum velocity does not occur when the head is a maximum. When the lag is very large, the maximum velocity occurs at a moment when the head is very small.

256. *"Hydraulic" or "frictional" flow.*—The first of the preceding examples (fig. 38) shows that if the head in a tidal channel is sufficient to produce strong currents, and the channel is not of great depth, the lag of the current with respect to the head is small, and the current at any instant is substantially the same as that which would be produced by the instantaneous head were the flow steady. The flow under these conditions is often termed "hydraulic." A better name is "frictional tidal flow." Currents of this character are found in the East River, N. Y., and in other tidal straits of moderate depths which are subject to a considerable tidal head.

If the lag is small, the value of $\sqrt{\cos\phi}$ in equation (155) is close to unity, and the amplitude, B, of the velocity varies from day to day as the square root of the amplitude, S, of the slope, and hence as the square root of the amplitude of the head, H, during the tidal cycle. Since the tides at the ends of a tidal strait keep in general step as their amplitudes change from day to day with the changing declinations and distances of the moon, the daily variation in H is nearly proportional to the daily variation in the tidal range. When therefore the flow in a strait is largely frictional, or "hydraulic," the "strength of the current" in each section of the channel varies from day to day approximately as the square root of the tidal range.

257. *Frictionless tidal flow.*—The lag of the current increases as the slopes in the channel and the current velocities decrease. It increases also as the depth of the channel and the coefficient C increase. As shown in the last example (fig. 40), the lag becomes very large in deep channels with small slopes. Most of the potential energy due to the head in the channel is then taken up in the acceleration and deceleration of the current and little in overcoming frictional resistance. The flow under these conditions is sometimes termed "tidal," as distinguished from the "hydraulic" flow determined principally by frictional resistance. A better name is "frictionless flow."

258. In a section of channel which is so deep, or in which the currents are so weak, that the flow is nearly frictionless, ϕ is nearly 90°. A small error in taking off its value from table IX would then produce a large error in the computation of the amplitude, B, from equation (160). When ϕ is large, the value of B is better derived from equation (156):

$$B=(g/a)S \sin \phi$$

For tidal fluctuations having the speed of the M_2 component, and for $g=32.16$, the value of g/a is 228,900; and its logarithm is 5.35958.

For wholly frictionless flow, $\phi=90°$, and

$$B=gS/a=gH/al \tag{164}$$

The amplitude of the fluctuations of the current then varies from day to day directly as the head, and hence nearly as the amplitude of the tide.

Obviously, in the fiords of Alaska, where depths of 1,000 feet are common, and in other deep channels such as the Florida straits, the tidal flow is essentially frictionless.

259. If the depths in a channel are small enough and the currents sufficiently marked to be of consequence to shipping, the tidal flow is not frictionless and the currents depend upon both the friction head and the acceleration head, as indicated in the second example (fig. 40). The so-called hydraulic state of flow is one of degree only, and merges without distinction into conditions of flow in which the acceleration head becomes of increasing importance. The maximum velocity, or the "strength of the current" is always less than that which would be produced by the maximum head were the flow steady. The acceleration head acts as a brake on the currents as the friction head diminishes.

DISTORTIONS OF PRIMARY CURRENT

260. The primary current has been derived by taking the surface slope as a simple harmonic fluctuation; dropping the velocity head term from the general equation of motion (equation 112); substituting for the friction term its principal harmonic component $(8/3\pi)\ Bv/C^2r$; and taking the hydraulic radius, r, and the Chezy coefficient, C, at mean tide. The corrections for these approximations will now be developed. These corrections produce a velocity-time curve which is more or less distorted from the simple harmonic curve of the primary current.

261. *Corrections for the variation of frictional resistance with the reversing square of the velocity.*—The corrections to fulfill the condition that the friction term is $\pm v^2/C^2r$, may be computed, to any desired degree of refinement, by a somewhat laborious process explained in detail in appendix II. As there shown, these corrections, designated as i, are proportional to the amplitude, B, of the primary current and depend upon its angular lag, ϕ, and its phase, $at+\beta$. The correction factors, i/B, as so computed for successive values of ϕ, and for values of $at+\beta$ from 0 to 180°, are shown in table X. For values of $at+\beta$ between 180° and 360° the table is entered with $at+\beta-180°$ and the algebraic sign of correction reversed. As will be seen from the table, the corrections are small when ϕ is large, and the flow consequently is nearly frictionless. They become zero when $\phi=90°$.

TABLE X.—*Correction factor i/B*

$\phi=$	0	5°	10°	20°	30°	40°	50°	60°	70°	80°
$at+\beta=0$	0	−0.18	−0.18	−0.16	−0.14	−0.11	−0.08	−0.05	−0.03	−0.01
5°	+.18	−.14	−.16	−.15	−.13	−.10	−.07	−.04	−.03	−.01
10°	+.21	0	−.09	−.12	−.11	−.08	−.06	−.04	−.02	−.01
15°	+.21	+.12	0	−.07	−.07	−.06	−.05	−.03	−.02	−.01
20°	+.20	+.16	+.08	−.01	−.04	−.04	−.03	−.02	−.01	0
25°	+.18	+.18	+.13	+.04	0	−.01	−.01	0	0	0
30°	+.15	+.17	+.14	+.07	+.03	+.01	0	+.01	0	0
35°	+.12	+.14	+.14	+.09	+.05	+.03	+.02	+.01	+.01	0
40°	+.10	+.11	+.12	+.10	+.06	+.04	+.03	+.02	+.01	+.01
45°	+.07	+.08	+.10	+.09	+.07	+.05	+.03	+.02	+.02	+.01
50°	+.04	+.06	+.07	+.08	+.06	+.05	+.03	+.03	+.02	+.01
55°	+.02	+.03	+.04	+.06	+.05	+.04	+.03	+.03	+.02	+.01
60°	−.01	0	+.02	+.03	+.04	+.03	+.03	+.02	+.02	+.01
65°	−.02	−.02	−.01	+.01	+.02	+.02	+.02	+.02	+.01	+.01
70°	−.05	−.04	−.03	−.01	+.01	+.01	+.01	+.01	+.01	+.01
75°	−.06	−.05	−.05	−.03	−.01	0	0	0	+.01	0
80°	−.07	−.07	−.06	−.04	−.03	−.01	−.01	0	0	0
85°	−.08	−.07	−.07	−.06	−.04	−.03	−.02	−.01	−.01	0
90°	−.08	−.08	−.08	−.07	−.05	−.04	−.03	−.02	−.01	−.01
95°	−.08	−.08	−.08	−.07	−.06	−.05	−.04	−.02	−.02	−.01
100°	−.07	−.07	−.08	−.07	−.06	−.05	−.04	−.03	−.02	−.01
105°	−.06	−.06	−.07	−.07	−.06	−.05	−.04	−.03	−.02	−.01
110°	−.05	−.05	−.06	−.06	−.06	−.05	−.05	−.03	−.02	−.01
115°	−.02	−.04	−.04	−.05	−.06	−.05	−.05	−.03	−.02	−.01
120°	−.01	−.02	−.03	−.04	−.05	−.05	−.04	−.03	−.02	−.01
125°	+.02	0	−.01	−.02	−.03	−.04	−.04	−.03	−.02	−.01
130°	+.04	+.03	+.02	0	−.02	−.03	−.03	−.02	−.02	−.01
135°	+.07	+.05	+.04	+.02	0	−.01	−.02	−.02	−.01	0
140°	+.10	+.08	+.07	+.04	+.02	0	−.01	−.01	−.01	0
145°	+.12	+.11	+.09	+.06	+.04	+.02	+.01	0	0	0
150°	+.15	+.13	+.12	+.09	+.06	+.04	+.02	+.01	+.01	+.01
155°	+.18	+.16	+.14	+.11	+.08	+.05	+.03	+.02	+.01	+.01
160°	+.20	+.18	+.16	+.13	+.10	+.07	+.05	+.03	+.02	+.01
165°	+.21	+.20	+.18	+.15	+.11	+.08	+.06	+.04	+.02	+.01
170°	+.21	+.21	+.19	+.16	+.13	+.09	+.07	+.05	+.03	+.01
175°	+.18	+.20	+.20	+.17	+.13	+.10	+.07	+.05	+.03	+.01
180°	0	+.18	+.18	+.16	+.14	+.11	+.08	+.05	+.03	+.01

262. *Examples.*—The primary current in a section of the Cape Cod Canal, at the time t after a lunar transit, as derived in paragraph 249, is:

$$v=5.27 \sin (m_2 t+124°02')$$

and its angular lag, ϕ, is 10°.2.

The corrected velocity at say 3 lunar hours after a lunar transit is to be found. When t is reckoned in lunar hours $m_2=30°$. Then, at the given time, $at+\beta=90°+124°02'=214°.03$. Since this angle lies between 180° and 360°, table X is entered with $\phi=10°.2$ and

$$at+\beta=214.03°-180°=34°.03$$

The corresponding value of i/B is, by interpolation, $+0.14$. Reversing the sign, and multiplying by $B=5.27$, the correction is -0.74 feet per second. The primary current at the given hour is

$$5.27 \sin 214°.02'=--2.95$$

and the corrected velocity is -3.69 feet per second.

263. The distortion of the primary current curve in this section of the canal, derived by applying the correction at successive lunar hours, is shown in figure 41. The distortion of the primary current curve in a section of the Delaware River (par. 251 and fig. 39), is shown in figure 42. In the latter case B is 1.96 feet per second, and ϕ is 36°.

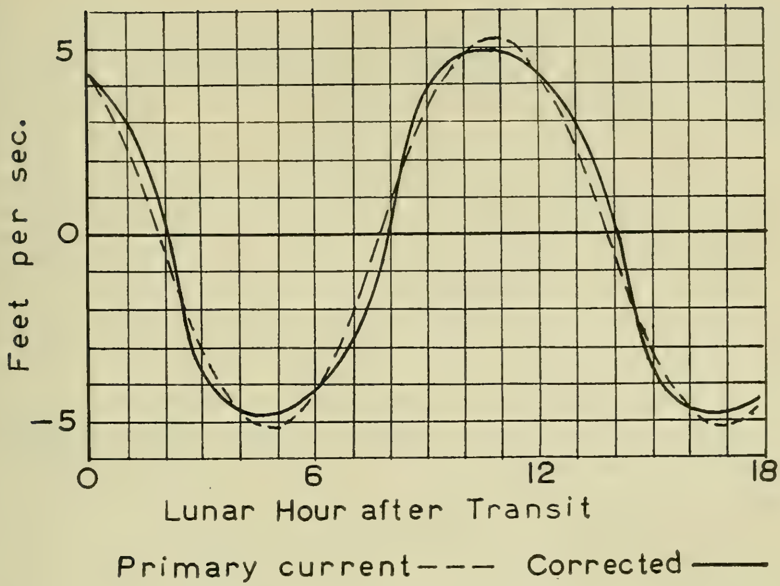

FIGURE 41.—Distortion of primary current when $\phi=10°.2$.

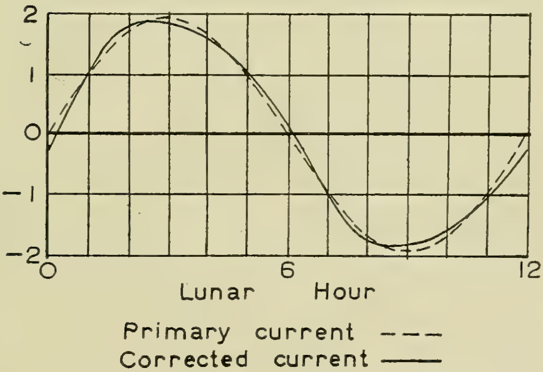

FIGURE 42.—Distortion of primary current when $\phi=36°$.

264. *Correction for remaining approximations.*—The correction of the primary current for the remaining approximations introduced in its derivation may be computed by a variation of the procedure by which the corrections shown in table X are derived. The computations determine the corrections at selected time intervals through the tidal cycle. In order that they may apply to repetitions of the cycle, the intervals should be parts of the component hour of the simple harmonic fluctuation of the head used for the determination of the primary current, ordinarily the lunar hour of 1.035 mean solar hours. Since the computations depend, in part, on the changes in the velocity during these intervals, their accuracy is increased, but the labor mul-

tiplied, as the interval is decreased. Intervals of half a lunar hour usually are sufficiently small to give acceptable determinations.

265. The corrected velocities must be such as to satisfy equation (131):

$$h_s+h_v+h_a+h_f=0$$

Let v be the velocity of the primary current, corrected by i, from table X, on a given lunar half hour,

δ, the further correction to v,
δ_0, the correction at the preceding half hour,
l, the length of the section.

266. For purposes of the computations, the surface head, h_s, should have a fluctuation which, although not necessarily a simple harmonic, identically repeats itself every 12 lunar hours if the tides and the surface head are wholly semidiurnal, or every 24 lunar hours if the diurnal components of the head are so large as to require consideration. This result may be accomplished by selecting tidal fluctuations at the ends of the section which identically repeat themselves every 12 or 24 lunar hours. Under ordinary circumstances it is indeed apparent that the tides on one day have but little effect upon the currents of the next.

267. The expression for the acceleration head is, from equation (129)

$$\begin{aligned} h_a &= (l/g)\,\partial(v+\delta)/\partial t \\ &= (l/g)\,(\partial v/\partial t+\partial\delta/\partial t) \end{aligned}$$

Since this relation remains approximately true when small finite increments are substituted for the differentials, it is permissible to place:

$$\begin{aligned} h_a &= (l/g)\,(\Delta v/\Delta t+\Delta\delta/\Delta t) \\ &= (l/g\Delta t)\,(\Delta v+\Delta\delta) \end{aligned} \qquad (165)$$

In which Δt is the selected time interval, in mean solar seconds, and Δv and $\Delta\delta$ are the increases in v and δ corresponding thereto at the given half hour.

268. It will be convenient to place:

$$l/g\Delta t=b \qquad (166)$$

When the time interval is a half lunar hour:

$$\Delta t=\tfrac{1}{2}\times 1.035\times 3{,}600 \text{ seconds}$$

and:

$$b=0.0000167l \qquad (167)$$

The values of Δv are computed from the successive values of v. As the best approximation, $\Delta\delta$ will be taken as the increase, $\delta-\delta_0$, in the preceding interval. Equation (165) then becomes:

$$h_a = b\Delta v + b(\delta-\delta_0) \qquad (168)$$

269. The friction head, h_f, is, from equation (130):

$$h_f = \pm l(v+\delta)^2/C^2r$$

in which C and r vary with the stage of the tide. A diagram may be prepared showing the values of:

$$F = l/C^2r \qquad (169)$$

corresponding to the stages of the tide.

The expression for the friction head may be written:

$$\begin{aligned} h_f &= \pm F(v+\delta)^2 \\ &= \pm Fv^2 \pm 2Fv\delta \pm F\delta^2 \end{aligned} \qquad (170)$$

Since δ is a comparatively small correction, at least a first approximation may be derived by dropping its square and neglecting any effect that it may have upon the algebraic sign of the corrected velocity, $v+\delta$; giving:

$$h_f = \pm Fv^2 + 2\delta(\pm v)F \qquad (171)$$

in which the positive sign is to be applied when v is positive, and the negative sign when negative. Obviously, therefore, the factor $(\pm v)$ is always positive. Representing the numerical value of v, on the given half hour, as $\bar{v}$, equation (171) becomes:

$$h_f = \pm Fv^2 + 2\delta F\bar{v} \qquad (172)$$

270. The velocity head, h_v, remains to be considered. The derivation of the tidal currents in a short section of channel was predicated on the assumption that the section is so short that at any instant the variation of the velocity between the ends of section is immaterial. Under this assumption the velocity head would disappear. While the derivation remains valid even though there be a sufficient difference between the velocities at the ends of the section to produce some velocity head, yet in the ordinary case it is too small to be worth computing. It may be included in the computations by determining the velocities at the ends of the section at the successive intervals of time. For this purpose the change in the discharge between the ends of the section because of the storage and release of water with the rise and fall of the tide, as developed in subsequent chapters, must be taken into consider-

ation as well as the cross section of the channel. In any case the effect of the correction δ upon the velocity head may be neglected.

271. Substituting in equation (131) the expressions derived for the several heads:

$$h_s+h_v+b\Delta v\pm Fv^2+b(\delta-\delta_0)+2\delta F\bar{v}=0$$

and, placing:

$$h_s+h_v+b\Delta v\pm Fv^2=-R \tag{173}$$

it becomes:

$$b(\delta-\delta_0)+2\delta F\bar{v}-R=0$$

whence:

$$\delta=(\delta_0+R/b)/(1+2F\bar{v}/b) \tag{174}$$

It should be observed that if v and Δv were the correct velocity and its increment for the given time, R would be zero. R is then the residual head which δ is to remove.

272. The computation of δ from equation (174) is most readily explained by applying it to the concrete example of the final adjustment of the currents produced by the average tides in the section of the Cape Cod Canal between stations 180+30 and 225, as of September–October 1932. In the computation, these tides are referred to a datum 10 feet below mean sea level, so that all tidal elevations are positive. At mean sea level, elevation 10, the hydraulic radius of the section has been taken as 22.7 feet, and C at 90 (par. 249). From the cross sections of the canal, the value of r at elevation 14 is found to be 24.6 feet, and at elevation 6 to be 20.9 feet. The corresponding values of C will be taken as 91 and 89, respectively. Since $l=4{,}470$, the values of $F=l/C^2r$ are:

Values of F			
Tide (y)	r	C	F
14	24.6	91	0.0219
10	22.7	90	.0243
6	20.9	89	.0270

A diagram prepared from this data gives the value of F for any value of y between 6 and 14.

The coefficient b, for intervals of a half lunar hour, is, from equation (167):

$$b=0.0000167\times4470=0.0746$$

The lag of the primary current has been found to be 10°12′, and its equation, with a lunar transit as the origin of time, to be (par. 249):

$$v=5.27\sin(\mathrm{m}_2t+124°02')$$

273. The computation of the residuals, R, may be made in the form illustrated in the following tabulation.

RESIDUALS

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
t	v	$v-v_0$	Δv	y_0	y_1	y	F	h_s	$b\Delta v$	$\pm Fv^2$	R
0	4.32	−0.38	−0.45	11.95	11.54	11.75	0.0233	−0.41	−0.03	+0.44	0
.5	3.80	−.52	−.63	11.06	10.76	10.92	.0238	−.30	−.05	+.34	+.01
1.0	3.06	−.74	−.90	10.09	9.94	10.01	.0243	−.15	−.07	+.23	−.01
.5	2.01	−1.05	−1.30	9.14	9.11	9.13	.0249	−.03	−.10	+.10	+.03
2.0	.47	−1.54	−2.04	8.21	8.35	8.28	.0255	+.14	−.15	+.01	0
.5	−2.06	−2.53	−2.08	7.42	7.70	7.56	.0259	+.28	−.16	−.11	−.01
3.0	−3.69	−1.63	−1.19	6.81	7.21	7.01	.0263	+.40	−.09	−.36	+.05
*		*		*		*		*		*	*
10.0	4.70	+.26	+.21	13.73	13.18	13.46	.0222	−.55	+.02	+.49	+.04
.5	4.85	+.15	+.08	13.63	13.06	13.35	.0223	−.57	0	+.52	+.05
11.0	4.85	0	−.08	13.28	12.72	13.00	.0225	−.56	0	+.53	+.03
.5	4.70	−.15	−.27	12.71	12.20	12.46	.0228	−.51	−.02	+.50	+.03
12.0	4.32	−.38	−.45	11.95	11.54	11.75	.0233	−.41	−.03	+.44	0

The time in lunar hours after a lunar transit is entered in column (1) and the primary current corrected by i, table X, in column (2). Column (3) shows the increase in the velocity during the preceding interval, the entry for 0 hour being repeated from the twelfth hour. The values of Δv, column (4) are the means of the entries for the given and the following half hours, and are consequently the average of the increments in v during the preceding and following intervals. The tidal elevations, y_0, at the initial end of the section, station 180+30, taken from the equation of the tide at this station, paragraph 240, are entered in column (5), and those at station 225 in column (6). The tidal elevation, y, in the section, column (7) is the average of the entries in columns (5) and (6). The corresponding values of F, from the diagram, are entered in column (8).

The surface heads, h_s, column (9) are the algebraic differences, y_1-y_0, from columns (6) and (5). The entries in column (10) are the products of Δv, column (4) times the constant $b=0.0746$. The slide rule affords satisfactory accuracy for these and the subsequent computations. The entries in column (11) are obtained by multiplying v^2, from column (2) by F, column (8) and have the algebraic sign of v. The residuals, R, column (12) are then the algebraic sum of the entries in columns (9), (10), and (11), with the sign reversed; the sums of the entries in columns (9), (10), (11), and (12) being zero.

The computation for hours 3.5 to 9.5 is not shown, but is carried out by the same procedure. In the example chosen, the tides at the ends of the section, and consequently the surface head, have in fact a simple harmonic fluctuation, but the computation would be in the same form if they had any other repeating fluctuation. The velocity head is omitted. If computed, it would be entered in an additional column and included in the derivation of R.

274. The computation of the corrections is completed in the following tabulation.

Corrections

(1)	(2)	(13)	(14)	(15)	(16)	(17)	(18)	(19)
t	v	R/b	$1+2F\bar{v}/b$	δ_0+R/b	δ	δ_0+R/b	δ	v'
0	+4.32	0	3.69	------	0	+0.14	+0.04	+4.36
.5	+3.80	+.13	3.42	+0.13	+.04	+.17	+.05	+3.85
1.0	+3.06	−.13	3.00	−.09	−.03	−.08	−.03	+3.03
.5	+2.01	+.40	2.34	+.37	+.16	------	------	+2.17
2.0	+.47	0	1.32	+.16	+.12	------	------	+.59
.5	−2.06	−.13	2.48	−.01	0	------	------	−2.06
3.0	−3.69	+.67	3.60	+.67	+.19	------	------	−3.50
.5	−4.44	+.94	4.16	+1.13	+.27	------	------	−4.17
4.0	−4.70	+.80	4.35	+1.07	+.25	------	------	−4.45
.5	−4.85	+.80	4.45	+1.05	+.24	------	------	−4.61
5.0	−4.85	+.80	4.42	+1.04	+.24	------	------	−4.61
.5	−4.70	+.54	4.26	+.78	+.18	------	------	−4.52
6.0	−4.32	+.54	3.95	+.72	+.18	------	------	−4.14
.5	−3.80	+.13	3.53	+.31	+.09	------	------	−3.71
7.0	−3.06	+.13	2.99	+.22	+.07	------	------	−2.99
.5	−2.01	−.40	2.28	−.33	−.13	------	------	−2.14
8.0	−.47	0	1.29	−.13	−.10	------	------	−.57
.5	+2.06	+.27	2.26	+.17	+.08	------	------	+2.14
9.0	+3.69	+.13	3.23	+.21	+.07	------	------	+3.76
.5	+4.44	+.27	3.65	+.34	+.09	------	------	+4.53
10.0	+4.70	+.54	3.80	+.63	+.17	------	------	+4.87
.5	+4.85	+.67	3.89	+.84	+.22	------	------	+5.07
11.0	+4.85	+.40	3.91	+.62	+.16	------	------	+5.01
.5	+4.70	+.40	3.87	+.56	+.14	------	------	+4.84
12.0	+4.32	0	3.69	+.14	+.04	------	------	+4.36

The residuals from column (12), divided by $b=0.0746$ are entered in column (13). The divisor, $1+2F\bar{v}/b$, of equation (174) is derived from the numerical value of $\bar{v}$, column (2) and of F, column (8), and entered in column (14). Since an initial value of δ_0 is not known, the computation of δ in columns (15) and (16) is started with an initial correction of zero at 0 hour. At 0.5 hour, $\delta_0+R/b=0+.13=.13$. Applying the divisor, column (14) the first determination of δ at 0.5 hour is 0.04. At 1.0 hour $\delta_0+R/b=0.04-0.13=-0.09$; and δ at 1.0 hour is -0.03. A continuation of the process gives $\delta=0.04$ at 12 hours. With this initial value the computation is repeated in columns (17) and (18) until, at 1 hour, the value of δ becomes that previously found. The adjusted velocities are then found, in column (19) by applying the final values of δ to the velocities in column (2). The velocities, before and after this final adjustment, are shown in figure 43.

275. *Discussion.*—The final corrections, δ, developed in the preceding example, are so small that the corrected velocities do not differ substantially from those used in their determination, and a recomputation is unnecessary. Since the surface head assumed in their derivation has a simple harmonic fluctuation, these corrections are due only to the variation of r and C with the rise and fall of the tide. Such corrections depend upon the relative timing of the tide and current and on the relation between the tidal range and channel depth. For a given tidal range and channel depth they are the greatest when,

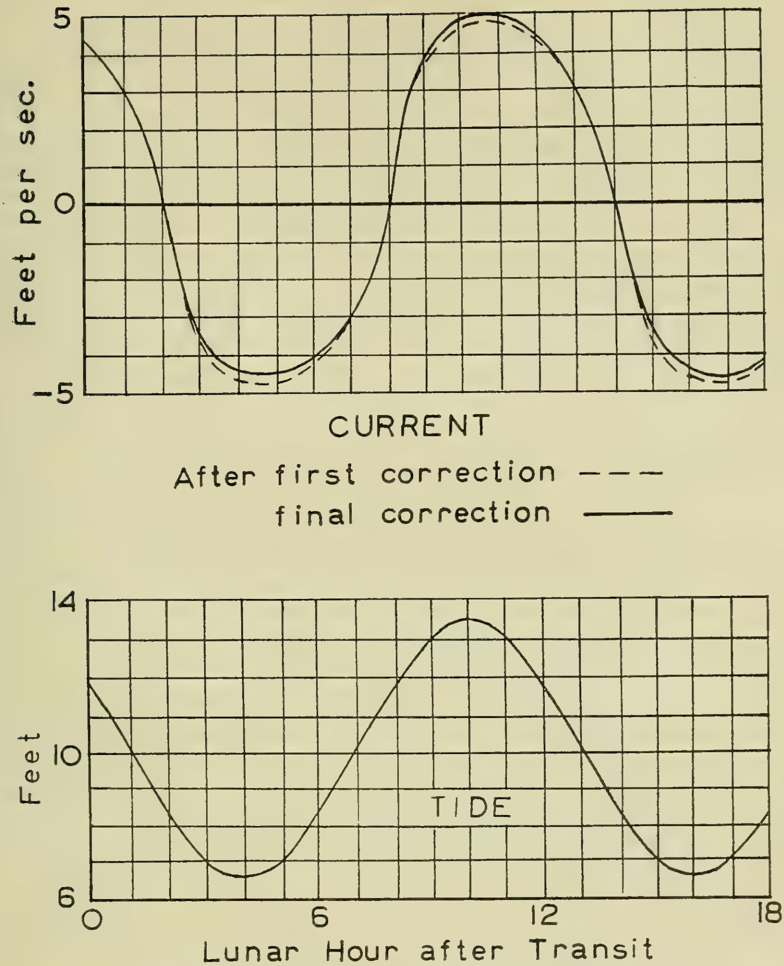

FIGURE 43.—Final adjustment of computed current in Cape Code Canal.

as in the example chosen, the timing is such that the maximum currents occur at high and low tide. They become smaller as the angular lag, ϕ, of the primary current increases and the frictional resistance consequently has a less effect upon the currents.

276. The residuals developed in the example chosen amount to but a few hundredths of a foot. Much larger residuals, and consequently much larger corrections to the velocity, would be produced if during the tidal cycle the surface head varied from a simple harmonic fluctuation by as much as a tenth of a foot. As will later be shown, considerable distortions of the surface heads in successive sections of a long tidal channel may be produced in the filling and emptying of the tidal prism at different tidal stages, and other distortions are produced by the diurnal components of the head. If the first computation devel-

ops comparatively large corrections to the velocity, the computations should be repeated with the corrected velocities derived from the first computation.

277. The relatively small distortions of the tides needed to produce a large distortion of the surface head and currents is illustrated in figure 44. The current there shown has the equation:

$$v = \sin m_2 t + \sin 3m_2 t$$

Neglecting the velocity head, the equation of motion becomes:

$$\partial y/\partial x + (m_2/g)(\cos m_2 t + 3\cos 3m_2 t) \pm (\sin m_2 t + \sin 3m_2 t)^2/C^2 r = 0$$

Whence:

$$h_s = (l\partial y/\partial x)$$
$$= -(lm_2/g)(\cos m_2 t + 3\cos 3m_2 t) \pm (l/C^2 r)(\sin m_2 t + \sin 3m_2 t)^2.$$

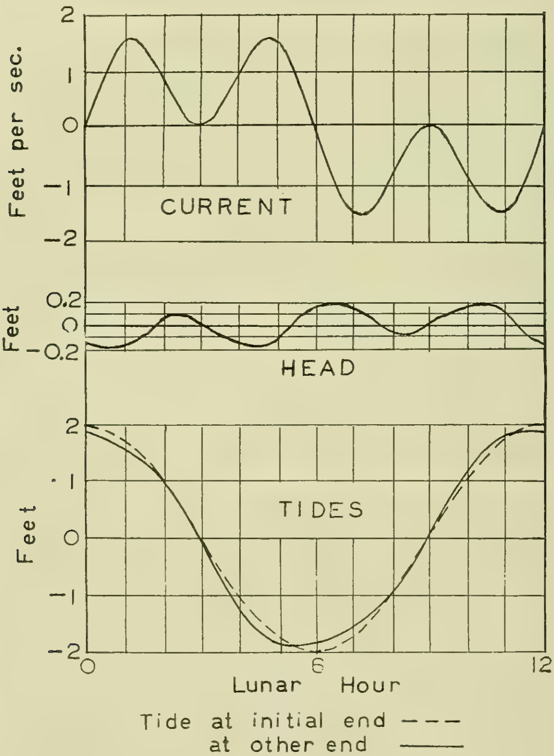

FIGURE 41.—Comparative distortions of current and tides.

The surface head, h_s, in a section 10,000 feet long, when $r=20$ and $C=90$, is plotted in the figure. Taking the tide at the initial end of the section as a simple harmonic fluctuation with an amplitude of

2 feet, the tide at the other end would be distorted only to the extent shown.

278. *Comparison with measured current curves.*—The form of the computed tidal current curves derived in the preceding paragraphs may be compared with curves of measured velocities, and with those derived by the method of cubature hereafter described. It should be recollected that the computed curves show the *mean* velocity at a cross section of the channel during a tidal cycle. A meter measurement of the mean velocity in a tidal channel is quite a difficult undertaking, as the velocity in each area of the cross section is changing continuously, while its fluctuations are not identically repeated from day to day. Available records often show the velocities only at a single point in the cross section; but these indicate the characteristic *shape* of the current curve. Current curves derived by the cubature of an estuary show, on the other hand, the mean velocities at the cross section.

279. The currents in the Cape Cod Canal afford a typical example of the form of the current curve when the flow is markedly frictional. The average measured midstream current velocities at 0.3 depth, at station 225, after the time of a lunar transit, compiled from a series made by the United States Engineer Department, September 28–October 6, 1938; and the corresponding mean tide curve in the section from station 180+30 to station 225, are shown in figure 45. The velocity curve has, it will be seen, the characteristics of the computed curve of mean velocities, shown in figure 43. As is to be expected, the midstream velocities are about 25 percent in excess of the mean velocities throughout the cross section.

280. The current in the estuary of the Delaware at the head o Delaware Bay, as determined by a mean cubature made by the United States Engineer Office in Philadelphia, shown in figure 49, page 154, affords an example of a typical velocity curve when the flow is of a less frictional character, and is not greatly modified by overtides. This curve may be compared with the curve of computed velocities shown in figure 42.

281. The marked effect of overtides on the currents is illustrated by the velocity curve at Philadelphia 63 miles further up the Delaware estuary, determined by the same cubature, and shown in figure 50, page 155.

282. The even greater distortions of the current in some tidal channels is illustrated by the curve of measured channel velocities in Seekonk River, R. I., shown in figure 46, page 141, taken from the Manual of Current Observations, United States Coast and Geodetic Survey.

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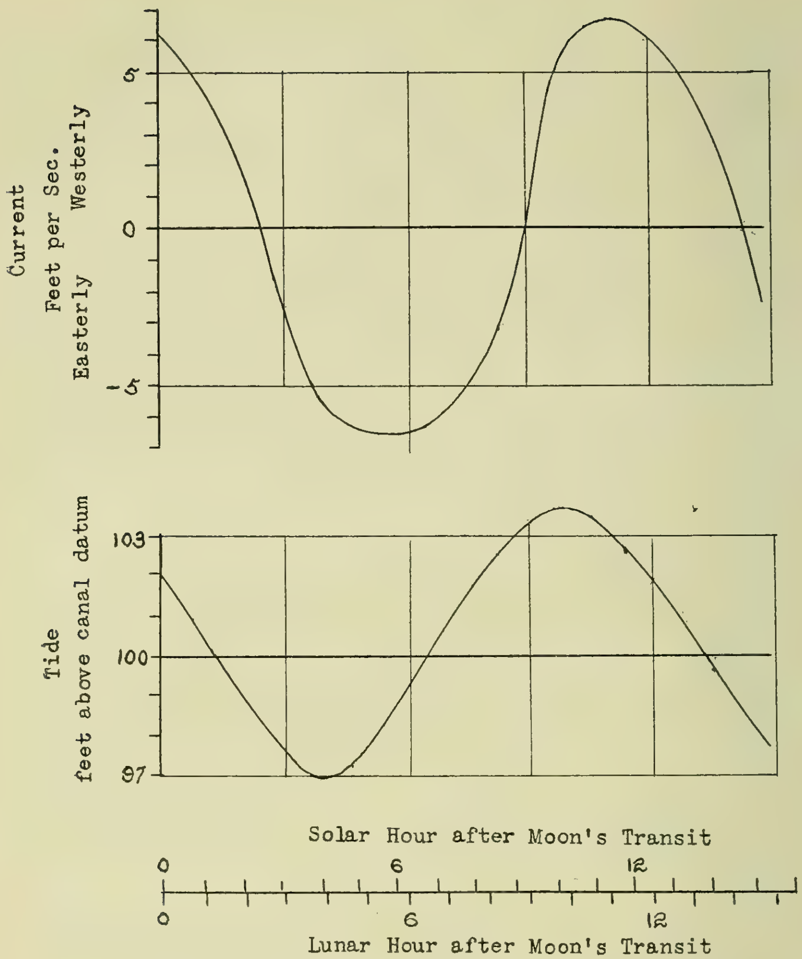

Average measured midstream velocity, 0.3 depth
Sta.225, Cape Cod Canal, Sept.28 - Oct.6, 1932.

FIGURE 45.

283. *Summary.*—The preceding formulas and examples show that the deformation of the primary current because of the variation of frictional resistance with the square of the velocity is not large unless the currents are unusually strong and the channel is of moderate depth, so that the flow is largely frictional. Its deformation because of the varying channel depth depends on the relation between the timing of the tide and the timing of the current, as well as on the ratio of the tidal range to the mean depth in the channel, and usually is quite small in deep channels. The deformation because of the effect of overtides may be quite large. If the ascertained variation in the surface

head in a short section of a tidal channel can be closely reproduced by a simple harmonic fluctuation, and the proper coefficient of friction selected, the computed primary current affords a fair representation of the actual currents, but a closer approximation would be secured by applying the corrections developed in the preceding paragraphs.

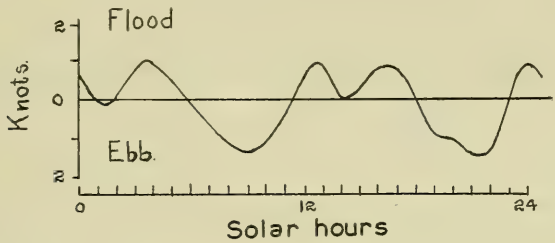

FIGURE 46.—Current in Seekonk River, R. I., showing effect of short-period constituents.

LIMITATIONS ON THE COMPUTATION OF CURRENT VELOCITIES FROM THE OBSERVED HEADS IN A SHORT SECTION OF TIDAL CHANNEL

284. As is well recognized, the velocity in a natural channel cannot be reliably determined from the observed head even when the flow is steady. In a short section of a tidal channel, the computation of the velocities from the observed heads presents further complications. These heads are the relatively small differences between the changing tidal heights at gaging stations at the two ends of the selected section of channel. Considerable accidental errors are inevitable in taking off the tidal heights from the somewhat irregular curve produced by a recording tide gage, and even greater errors in the timed readings of a staff gage. When these departures happen to be in opposite directions they produce errors which are large in proportion to the head. The heads derived from the differences of observed hourly readings are apt, therefore, to vary so erratically as to afford little basis for a determination of the velocities. The most workable procedure is to find the harmonic fluctuations which most nearly represent the actual fluctuations of the tides at the ends of the section, and to derive therefrom the corresponding harmonic fluctuation of the head. Obviously, more consistent results may be secured from average tide curves than from observations made during 1 day.

285. In a long tidal channel, the heads between the entrances usually are so large that accidental errors in the observed tidal heights at these entrances become of minor importance; but in such a channel the currents may be due more to the storage and release of water in the tidal prism than to the head between the entrances.

286. In short, a direct measurement of the actual velocities in a channel, however crude, is more reliable than the most refined calculation from the varying head and an assumed coefficient of roughness.

The relation between the surface head and the velocity in a short section of a channel, derived in this chapter, affords, however, a basis for estimating the currents in a projected long canal, by a procedure which is developed in detail in chapter VIII.

CURRENTS IN A SHORT SECTION OF CHANNEL WHEN THE FRICTIONAL RESISTANCE IS NEGLIGIBLE

287. If a channel is so deep, and the current velocities are so small, that the flow is essentially frictionless (par. 257), the currents produced in a short section of the channel by any fluctuation of the tides at the ends of section have a simple relation to the amplitudes and speeds of the harmonic components of these tides. Designating the amplitudes of the several harmonic components of the tide at the initial end of the section as M_2', S_2', etc., and at the other end as M_2'', S_2'', etc., the equation for the tide at the initial end becomes:

$$y_0 = M_2' \cos (m_2 t + \alpha_1') + S_2' \cos (s_2 t + \alpha_2') + \cdots$$

and at the other end:

$$y_1 = M_2'' \cos (m_2 t + \alpha_1'') + S_2'' \cos (s_2 t + \alpha_2'') + \cdots$$

The surface head through the section is then:

$$\begin{aligned} h_s = y_1 - y_0 = M_2'' \cos (m_2 t + \alpha_1'') - M_2' \cos (m_2 t + \alpha_1') \\ + S_2'' \cos (s_2 t + \alpha_2'') - S_2' \cos (s_2 t + \alpha_2') \\ + \text{etc.} \end{aligned} \quad (175)$$

288. Since the respective pairs of components of the same speed unite into components of that speed, equation (175) reduces to one in the form:

$$h_s = H_1 \cos (m_2 t + H_1°) + H_2 \cos (s_2 t + H_2°) + \cdots$$

In which the amplitudes, H_1, H_2 and the phases $H_1°$, $H_2°$ of the component surface heads could be computed by the process indicated in paragraph 239.

When both the velocity head term and the friction term in equation (112) are dropped, this equation becomes:

$$\partial y/\partial x + (1/g)\partial v/\partial t = 0 \quad (176)$$

Whence:

$$v = -g \int (\partial y/\partial x) \partial t \quad (177)$$

And in a section of channel so short that the change in slope in the section is negligible:

$$\partial y/\partial x = h_s/l = (H_1/l)\cos(\mathrm{m}_2 t + H^\circ) + (H_2/l)\cos(\mathrm{s}_2 t + H_2^\circ) + \cdots$$
$$= I_1 \cos(\mathrm{m}_2 t + H_1^\circ) + I_2 \cos(\mathrm{s}_2 t + H_2^\circ) + \cdots \quad (178)$$

289. The slopes, I_1, I_2, etc., in this equation quite evidently approach definite limits as the length, l, of the section is reduced.

Substituting in equation (177) and integrating:

$$v = -g \int (\partial y/\partial x)\,\partial t$$
$$= -(I_1 g/\mathrm{m}_2)\sin(\mathrm{m}_2 t + H_1^\circ) - (I_2 g/\mathrm{s}_2)\sin(\mathrm{s}_2 t + H_2^\circ) - \cdots + K \quad (179)$$

The constant of integration, K, is readily interpreted as an adventitious constant current through the channel, apart from the currents due to tidal fluctuations, and may be disregarded.

If then the flow in a tidal channel is essentially frictionless, the velocity of the current at any point in the channel is the resultant of component velocities with the speeds of the tidal components.

290. The inference should not be drawn from equation (179) that the amplitudes of the components of the velocity are proportional to the ratios of the amplitudes of the tidal components to their respective speeds; for the component heads and slopes, from which the velocities are derived, are determined by the *changes* in the amplitudes and phases of the tidal components at successive points along the channel, and not by the magnitude of these amplitudes.

COMPONENT CURRENTS

291. As shown in paragraph 289, when the tidal flow is essentially frictionless the current may be resolved into component currents, fluctuating at the same speeds as those of the tidal components. If the flow is not frictionless, each fluctuation of a tide of the semidiurnal type has been shown to produce a primary current with a simple harmonic fluctuation, to which minor corrections are to be applied. The amplitude of the primary current must vary from day to day with the variation in the amplitude of the resultant tide. The primary current should then be resolvable into components of fixed amplitudes, with the speeds of the tidal components. The corrections to the primary current, and its distortions due to overtides, are repeated almost identically in each successive tidal fluctuation, and are then reproduced by *overcurrents* whose speeds are integral multiples of the speeds of the principal tidal component.

292. Further minor current components are to be anticipated because of the variation of the friction term with the square of the velocity; for, if the primary current components are:

$B \sin(\mathrm{m}_2 t+\beta)$, $B_1 \sin(\mathrm{s}_2 t+\beta_1)$, and so on, the friction term becomes:

$$F = \pm [B \sin(\mathrm{m}_2 t+\beta) + B_1 \sin(\mathrm{s}_2 t+\beta_1) + \ . \ . \ . \ ]^2/C^2 r$$

$$= \pm (B^2/C^2 r) \sin^2 (\mathrm{m}_2 t+\beta) \pm (B_1^2/C^2 r) \sin^2 (\mathrm{s}_2 t+\beta_1) \ . \ . \ . \qquad (180)$$

$$\pm (2BB_1/C^2 r) \sin(\mathrm{m}_2 t+\beta) \sin(\mathrm{s}_2 t+\beta_1) \ . \ . \ .$$

The terms in this expression for F which contain the squares of the sines of functions of the speeds m_2, s_2, etc., afford components of the friction term with speeds of the corresponding harmonic components, and their overtides. The terms which contain the products of the sines of functions of these speeds may be replaced by the algebraic sum of the proper trigonometric functions of the sums and difference of the angles, and hence of the speeds. Components of the friction term, and corresponding components of the current, with speeds which are the sums and differences of the speeds of the principal tidal components, may therefore be anticipated. These may be termed *compound* current components.

293. The currents set up by tides of the mixed or of the diurnal types should equally well be resolvable into components with the speeds of the harmonic tidal components, together with overcurrents and compound current components. Furthermore, in the propagation of the tide through a long channel, the overcurrents and compound currents may create corresponding overtides and compound tides.

294. The mathematical relation between the components of the tide and the componenets of the current, when frictional resistance must be taken into consideration, does not appear to offer a profitable field for investigation; but, as explained in chapter X, the component currents may be determined by an harmonic analysis of the observed currents in a channel.

Chapter VI

CONTINUITY OF FLOW IN LONG TIDAL CHANNELS

DEFINITIONS

295. A few definitions may simplify the ensuing discussion:

The *tidal prism* of a channel is the prism between low water and high water.

A *long* tidal channel is one of such length that the filling and emptying of the successive sections of its tidal prism affects, more or less profoundly, the tidal currents and the tidal heights through the channel.

A *connecting* tidal channel connects two tidal seas. In a long connecting channel the tides and tidal currents through the channel are caused both by the surface head between the tides at the entrances and by the storage and release of water in its tidal prism. As a special case, a connecting channel may join a tidal with a tideless body of water. A natural connecting channel is usually termed a *strait*, and a short connecting channel leading from the ocean to a tidal or tideless bay or sound is termed an *inlet.*

A *closed* tidal channel leads inland from a tidal sea and terminates in a dead end. Its tides and currents are due solely to the filling and emptying of its tidal prism, together with the discharge of any flow which may enter the channel from the uplands.

A tidal *canal* is an artificially excavated tidal channel of regular dimensions.

EQUATION OF CONTINUITY

296. *Equation of continuity for steady flow.*—Let:

X be the area of a cross section of a channel,

Q the quantity of water passing through the cross section in a unit of time; designated as the *discharge* at the section.

v the mean velocity of the current at the section. Then obviously:

$$v = Q/X \tag{181}$$

In steady flow, Q is by definition the same constant at all cross sections, and equation (181) affords a complete expression of the condition of continuity of flow.

297. *Equation of continuity for tidal flow.*—In tidal flow, water is stored and released throughout a channel as the tide rises and falls, and Q therefore varies from section to section as well as varying at each section with the time.

Let S_0, figure 47, be a cross section of a tidal channel at a distance x from the point chosen as the origin of distance, and S_1, an adjacent

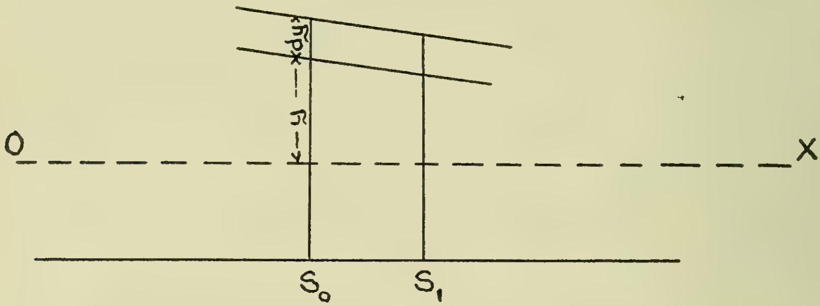

FIGURE 47.

cross section at the elementary distance dx from S_0. At section S_0, and at the time t, let:

- z be the surface width of the channel,
- X the area of the cross section,
- $D = X/z$ its mean depth,
- y the elevation of the water surface above any assumed horizontal plane of reference,
- $\partial y/\partial t$ the rate at which y is increasing with the time,
- v the mean velocity in the cross section,
- Q the discharge.

The volume of water passing S_0 during the elementary time interval dt is then Qdt. During the same interval the water surface between S_0 and S_1 rises the distance $(\partial y/\partial t)dt$. The volume of water passing S_1 during the interval is then decreased by the contents of the prism whose width is z, whose length dx and whose height is $(\partial y/\partial t)dt$. Designating the rate of decrease in discharge with the distance as $-\partial Q/\partial x$, the decrease in the discharge in the distance, dx, between the sections, is $-(\partial Q/\partial x)dx$, and the decrease in volume of water passing section S_1 in the time dt is $-(\partial Q/\partial x)dxdt$. Obviously, therefore:

$$-(\partial Q/\partial x)dxdt = zdx(\partial y/\partial t)dt$$

whence:

$$\partial Q/\partial x + z\partial y/\partial t = 0 \tag{182}$$

Equation (182) is the general equation of continuity in a tidal channel.

298. Since $Q=vX=vzD$, equation (182) may be written:

$$\partial(vzD)/\partial x+z\partial y/\partial t=0 \tag{183}$$

If a channel is of both constant width and constant depth below mean tide level, and the tidal fluctuation is so small with respect to the depth that the variation in D may be neglected, equation (183) becomes:

$$zD\partial v/\partial x+z\partial y/\partial t=0$$

or:

$$D\partial v/\partial x+\partial y/\partial t=0 \tag{184}$$

299. *Distinction between mean depth and hydraulic radius.*—Channels, whether natural or artificial, are usually so wide with respect to their depth that, if the tide does not overflow the banks of the channel proper, the mean depth, D, in the equation of continuity does not differ materially from the hydraulic radius, r, in the friction term of the equation of motion. On the other hand, if the channel is bordered by tide flats and sloughs, in which water is stored and released as the tide rises and falls, but which carry no appreciable current, the value of D may be much less than the value of r. In other words, D is computed from the gross width of the channel and r from the net width after deducting areas which carry no substantial flow.

CUBATURE OF A CLOSED CHANNEL

300. *Method of cubature.*—The currents in a closed tidal channel are caused by the filling and emptying of the tidal prism, and by the fresh-water discharge from any rivers and streams which may enter it. By taking simultaneous readings of the height of the tide at a sufficient number of stations between a given tidal station and the head of tide, the changes in the volume of water in the tidal prism from hour to hour, or at shorter intervals, may be computed, and the positive and negative discharges at the station due to the filling and emptying of the tidal prism ascertained therefrom. The total discharge is then the algebraic sum of the tidal discharge and the measured or estimated upland discharge. The mean velocity at the station at any given time may be determined by dividing the total discharge by the area of the cross section at the station at that time. This process is termed the *cubature* of the channel. It is essentially the arithmetic integration of the general equation of continuity.

301. *Basic data.*—The tidal stations established for a cubature should be spaced at such distances that no material error is introduced by taking the water surfaces between them as planes. This

condition ordinarily will be met by stations spaced some miles apart, but suitably placed with respect to marked changes in the cross section of the channel. The stations used in the cubature of the Delaware River by the United States Engineer Office at Philadelphia are as follows:

Station	Distance from head of tide in statute miles	Station	Distance from head of tide in statute miles
1. Trenton municipal pier	0. 45	11. Fort Mifflin	41. 67
2. Trenton marine terminal	1. 99	12. Baldwins	48. 45
3. Bordentown	5. 40	13. Marcus Hook	53. 83
4. Fieldsboro	6. 25	14. Edge Moor	60. 42
5. Florence	11. 08	15. New Castle	67. 52
6. Burlington	15. 32	16. Reedy Point	75. 15
7. Beverly	18. 60	17. Reedy Island	78. 88
8. Torresdale	23. 58	18. Artificial Island	83. 62
9. Delair	29. 11	19. Woodland Beach	92. 48
10. Philadelphia	33. 20	20. Ship John	97. 33

It may be necessary to establish as well tidal stations on any long tidal tributaries which enter the channel. For the convenience of the computations the tide of all stations should be referred or reduced to the same horizontal datum, preferably taken so low that all tidal heights are positive.

A reliable contour map is needed to show the tidal areas from low water to high water and measurements must be made of the cross sections of the channel at the stations where the velocities are to be determined.

302. A tidal channel whose cubature is to be made usually is the tidal portion of a river with a considerable drainage area. In the United States, gaging stations with established rating curves have been established above the head of tide on most rivers, and the upland discharge of the main stream, and of any important tributaries which enter the tidal section can be ascertained therefrom. If satisfactory rating curves have not been established, meter measurements should be made, at suitable stations above the head of tide, of the upland inflow from the main stream and any important tributaries entering the tidal section. The discharge from other drainage areas into the tidal channel, including those below the gaging stations, is relatively so small that it ordinarily can be derived with sufficient accuracy by estimating, from general data, the run-off per square mile.

303. *Selection of representative tides for cubature.*—The process of cubature would become an overwhelming task if repeated through the tides occurring during a considerable number of days. If the tides are of the semidiurnal type, with no great variation in range during the month, the cubature of the tides on a single day, chosen almost at random, will develop the characteristic fluctuations in the discharge and in the velocity at stations along the channel. The effect of the

diurnal inequality may be ascertained by extending the cubature through two semidiurnal tidal cycles; i. e., through a period of 25 hours. If the difference between spring and neap tides is large, a cubature might be made of a representative tide of each kind. If the tides are of the mixed or diurnal types, a cubature of a representative tropic tide and of a representative equatorial tide would be necessary to determine the characteristic currents produced by each.

304. *Average tide curves.*—A cubature based on average tide curves gives a better general picture of the discharges and currents in a tidal channel than one based on the tides during a single day. Cubatures prepared from average tide curves before and after a major change has been made in a channel afford a conclusive determination of the effect of the change upon the tidal discharge and currents. Average curves of tides of the semidiurnal type may be prepared by averaging the tidal heights, taken from the graphic record of an automatic tide gage, at hourly or half-hourly intervals for the 12 hours beginning with the time of each lunar transit. The observations should extend over a period of 15 or 29 days, or a multiple of the latter. A consideration of the principles of harmonic analysis, explained in chapter II, indicates that an average curve so prepared is substantially that of the M_2 component of the tide and its overtides.

Average curves of spring and neap tides may similarly be prepared by averaging the recorded tidal heights at hourly or half-hourly intervals after the lunar transit immediately preceding the times of spring and neap tides respectively; and average curves of tropic or equatorial tides by averaging the heights at the same intervals for a period of 25 hours after the lunar transits next preceding the times of such tides. Obviously, a long continuous record of the tides at each of the stations must be available to prepare good averages of tides which occur but twice a month.

305. *Composite curves of mean tidal fluctuations.*—The range of an average curve of all semidiurnal tides, prepared by the process outlined in the preceding paragraph, is less than the actual mean tidal range during the period. For the mean cubatures of the Delaware River made by the United States Engineer Office at Philadelphia, tide curves were prepared by computing, by the ordinary methods, the elevations and lunitidal intervals of mean low and high water, and connecting them with a composite curve derived from 10 recorded tide curves whose range, duration of rise and fall, and half-tide level were nearly the same as the range, duration of rise and fall and half-tide level of the mean tide. The composite curve is prepared by adjusting, proportionally, the duration and height of the rise and of the fall of each of the recorded tides to the mean duration and mean rise and fall, and averaging the results. For this purpose the periods from low water to high water and from high water to low water on

each of the recorded curves are divided into, say, 10 equal intervals, the recorded rise or fall during each interval is multiplied by the ratio of the total mean rise or fall to the total recorded rise or fall, and the results added successively to the computed elevation of low water. The tides at the proportional intervals are then averaged, and plotted at the corresponding mean intervals. The tides at semihourly intervals after a lunar transit then may be taken off the plotted curve. The result is a tide whose high water and low water are at the times and elevations of mean high and low water, and whose semihourly rates of rise or fall are the composite of those of the selected tides. This composite tide curve has a total period, from high water to high water, or from low water to low water, of half of a mean lunar day, 12.42 mean solar hours.

306. Similar composite curves of spring or neap tides could be prepared by adjusting a number of tides near the time of spring or neap tides to the computed times and elevations of mean low and mean high water of spring tides; and composite curves of tides of the mixed type by similarly adjusting suitable recorded tides to the times and elevations of mean lower low, higher low, lower high, and higher high waters. It may be observed that the sum of the durations of the rise and fall of spring and neap tides differs slightly from the mean lunar half day or day.

307. *Computations.*—Designating the successive tidal stations along the channel, beginning at or near the head of tide, as station 0, station 1, station 2 · · · station N, let:

$y_0, y_1, y_2, \cdots y_n$ be the heights of the tide at these stations at the time t, this time usually being on the hour and half hour.

Δt, the time interval used in the computations, usually ½ hour, or 1,800 seconds.

$y_0', y_1', y_2' \cdots y_n'$, the tidal heights at the time $t+\Delta t$.

$U_1, U_2, U_3 \cdots U_n$, the mean area of the water surface between stations 0 and 1, 1 and 2, etc., during the time interval between t and $t+\Delta t$.

$\Delta y_1, \Delta y_2, \Delta y_3 \cdots \Delta y_n$, the mean rise in the water surface between the successive stations during the same interval.

$\Delta V_1, \Delta V_2, \Delta V_3 \cdots \Delta V_n$, the algebraic increase in the volume of water between the successive tidal stations during the same interval.

Then evidently

$$\Delta V_1 = U_1 \Delta y_1,\ \Delta V_2 = U_2 \Delta y_2,\ \cdots,\ \Delta V_n = U_n \Delta y_n$$

If the stations are sufficiently close together, the mean rise in the water surface between any two stations during the time interval Δt may be taken as the increase in the mean elevation of the tides at the two stations during that period so that:

$$\Delta V_1 = U_1[(y'_0+y'_1)/2-(y_0+y_1)/2],$$
$$\Delta V_2 = U_2[(y'_1+y'_2)/2-(y_1+y_2)/2], \text{ etc.} \qquad (185)$$

The total increase in the volume of water in the tidal prism from the head of tide to any station, N, is then:

$$\Sigma\Delta V = \Delta V_1 + \Delta V_2 + \cdot \cdot \cdot, \Delta V_n \qquad (186)$$

This summation obviously should include the increase in volume in any long tidal tributary above station N which is separately cubatured.

Taking the origin of distances at the entrance to the channel, the tidal discharge and the velocity at a given station are positive during the tidal flood currents, when the total volume of tide water between the station and the head of tide is increasing, and negative during the ebb, when this volume is decreasing.

The mean tidal discharge, Q_t, at station N, during the time interval Δt, is then:

$$Q_t = \Sigma\Delta V/\Delta t \qquad (187)$$

The fresh water discharge, Q_f, at the station is similarly the sum of the fresh-water discharges entering the channel above that station. This discharge may be regarded as constant during the period of cubature. Since it is an outward discharge, it is intrinsically negative. The total discharge, Q, is then

$$Q = Q_t - Q_f \qquad (188)$$

Designating the mean area of the cross section at station N during the interval Δt as X, the mean velocity during this interval is:

$$v = Q/X \qquad (189)$$

308. The mean areas U_1, U_2, etc., of the water surface between the tidal stations during the successive intervals may be derived by taking off with a planimeter, from a map of the waterway, the areas at successive stages of the tide, and constructing a diagram from which the area at any elevation may be read. Ordinarily, it is sufficient to take off from the map the areas at high and low water and to join them on the diagram with a straight line. The area at each semi-hourly interval is read from the diagram at the mean elevation of the mean of the tides at the ends of the section.

The areas between the stations should include any tidal tributaries which enter the section, and should extend to the head of tide in these tributaries, unless tidal volumes in the tributary are cubatured from stations thereon. They should include also the effective storage area in any tidal marshes adjacent to the channel.

The cross section areas, X, at the tidal stations at which discharges and velocities are to be computed, similarly may be read from a diagram, constructed by taking off with a planimeter the area of the cross section at high and low water, and at intermediate elevations if necessary.

309. *Form for computations.*—The computations proceed from the head of tide downstream. A convenient form, developed for the cubature of the Delaware River, is shown in figure 48. To briefly illustrate the process, the computations for six half-hourly intervals only in the first two reaches below the head of tide and in the lowest reach, are entered on the same sheet. In the actual computation a separate sheet is used for each successive reach between the tide stations. The computation shown for the first reach is abbreviated as explained in paragraph 310.

The times and the heights of the tides at the upper and lower stations, selected for the cubature as explained in paragraphs 303 to 306 are entered in columns (1), (2), and (3), and the mean of columns (2) and (3) is entered in column (4). Column (5) designates the interval to which the entries in the succeeding columns apply. Column (6) is the mean of the given and preceding entries in column (4) and is therefore the mean elevation, during the interval, of the mean tides in the section. Column (7) is the surface area in the reach at the elevation shown in column (6). Column (8) is the algebraic increase in the entries in column (4) during the interval. The product of columns (7) and (8) is the value of ΔV for the interval (equation 185); entered in column (9). The increases, during the interval, in the tidal volumes of any separately cubatured tidal tributaries which enter the reach are inserted in columns (10) and (11). The total tidal volume, column (12), is the sum of columns (9) to (11). The total increase in volume during the interval in the upstream reaches, as previously computed for these reaches, is entered in column (13). The addition of the increase in the reach, column (12), gives the total increase, $\Sigma\Delta V$, at the lower station (column 14). The division by $\Delta t = 1{,}800$ seconds, gives the mean discharge during the period, column (15), and the addition of the fresh-water inflow (with the negative sign) give the total discharge, column (16). The mean elevation of the tide at the lower station, column (17), is the mean of the given and preceding entries in column (3). The corresponding area of the cross section at this station is entered in column (18); and the mean current velocity during the interval, column (19), obtained by dividing the entries in column (16) by those in column (18).

310. Because of the steady increase in the width of nearly all natural tidal channels from the head of tide to the entrance, the increases in the tidal volume between the stations near the head of the estuary are relatively very small. The upstream station may therefore be placed below the head of tide, and the successive values of ΔV at this

MEAN CUBATURE OF DELAWARE RIVER

Tides: Time	Tides: Upper Station	Tides: Lower Station	Tides: Average	Tidal Storage, Between Stations, Main Stream: ½ Hr. Interval ending at	Average Tide	Surface Area U	Change in Tide	Increase in Tidal Volume	Tributaries		Total (ΔV)	Total: To Upper Station	Total: To Lower Station (ΣΔV)	Discharges and Velocities at Lower Station: Tidal Discharge Qt	Total Discharge (+Fresh Water) Q	Mean Tide	Area of Cross Section X	Current Velocities
Hrs.	Ft.	Ft.	Ft.	Hrs.	Ft.	1000's Sq. Ft.	Ft.		Thousands		of Cubic	Feet		C.F.S.	C.F.S.	Ft.		Ft./Sec
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
				First Reach – Head of tide to Trenton Municipal Pier – Fresh water flow 12,000 cfs.														
3.5		3.93																
4.0		3.53		4.0	3.78	2,612	-0.40	-1,040					-1,040	-580	-12,580	3.73	6,610	-1.90
4.5		3.14		.5	3.34		-0.39	-1,020					-1,020	-570	-12,570	3.34	6,310	-1.99
5.0		2.78		5.0	2.96		-0.36	-940					-940	-520	-12,520	2.96	6,010	-2.09
5.5		2.43		.5	2.60		-0.35	-910					-910	-510	-12,510	2.60	5,750	-2.18
6.0		2.09		6.0	2.36		-0.34	-890					-890	-490	-12,490	2.26	5,500	-2.27
6.5		1.79		.5	1.94		-0.30	-780					-780	-430	-12,430	1.94	5,260	-2.36
				Second Reach – Trenton Municipal Pier to Fieldsboro – Fresh water flow 12,200 cfs.														
3.5	3.93	3.67	3.80															
4.0	3.53	3.28	3.40	4.0	3.60	41,890	-0.40	-16,760				-1,040	-17,800	-9,890	-22,090	3.48	21,390	-1.03
4.5	3.14	2.90	3.02	.5	3.21	41,000	-0.38	-15,620				-1,020	-16,640	-9,240	-21,440	3.09	20,970	-1.02
5.0	2.78	2.52	2.65	5.0	2.84	40,340	-0.37	-14,490				-940	-15,870	-8,820	-21,020	2.71	20,580	-1.02
5.5	2.43	2.16	2.30	.5	2.48	39,610	-0.35	-13,860				-910	-14,770	-8,210	-20,410	2.34	20,190	-1.03
6.0	2.09	1.81	1.95	6.0	2.12	38,880	-0.35	-13,610				-890	-14,500	-8,060	-20,260	1.98	19,800	-1.03
6.5	1.79	1.50	1.64	.5	1.80	38,220	-0.31	-11,850				-780	-12,630	-7,020	-19,220	1.66	19,480	-0.99
				Last Reach – Woodland Beach to Ship John Light – Fresh water Flow 19,900 cfs.					Cohanzy R.									
3.5	0.82	0.55	0.68															
4.0	0.52	0.42	0.47	4.0	0.58	696,600	-0.21	-146,300	-22,600		-168,900	-1,329,200	-1,498,100	-832,300	-852,200	0.48	516,300	-1.65
4.5	0.40	0.53	0.46	.5	0.46	696,400	-0.01	-7,000	-13,700		-20,200	-1,131,900	-1,152,100	-640,100	-660,000	0.48	516,300	-1.28
5.0	0.53	0.98	0.76	5.0	0.61	696,600	+0.30	+209,000	-1,700		+207,300	-836,400	-629,100	-349,500	-369,400	0.76	525,300	-0.70
5.5	1.11	1.58	1.34	.5	1.05	697,100	+0.58	+404,300	+8,800		+413,100	-348,500	+64,600	+35,900	+16,000	1.28	542,300	+0.03
6.0	1.78	2.27	2.02	6.0	1.68	697,900	+0.68	+474,600	+18,700		+493,300	+267,900	+761,200	+422,900	+403,000	1.92	564,000	+0.71
6.5	2.50	2.94	2.72	.5	2.37	698,800	+0.70	+489,100	+31,900		+521,000	+807,200	+1,328,200	+737,900	+718,000	2.60	586,600	+1.22

FIGURE 48.

station determined by multiplying the area U, between this station and the head of tide, by the half-hourly increases in the tide at the station, as shown in tabulated computations in figure 48.

311. *Graph of discharges and velocities.*—The fluctuations in the discharge and in the velocity at the successive tidal stations during the tidal cycle, and the relative importance of the tidal and fresh-water discharges, are made apparent by plotting the tidal heights, discharges

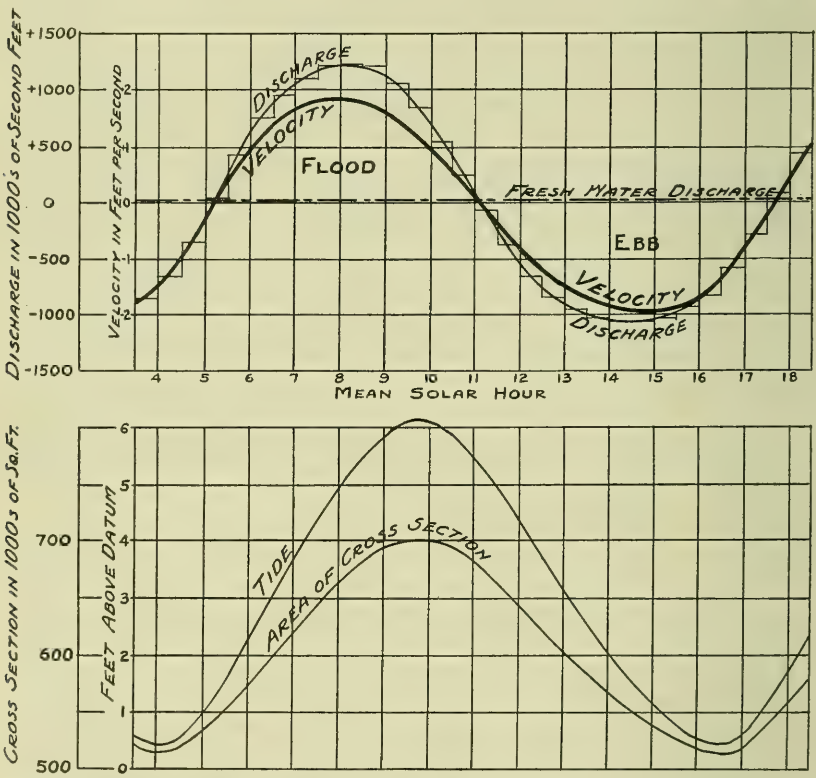

FIGURE 49.—Discharges and velocities at Ship John Light, Delaware Bay, from mean cubature.

and velocities through the tidal cycle. The discharges and velocities at Ship John Light, near the head of Delaware Bay, and at Philadelphia, 64 miles up the estuary of Delaware River, as derived from the mean cubature of this channel, are shown in figures 49 and 50.

312. The tidal discharges computed in the cubature of a channel (column 15 of fig. 48) are the mean discharges during successive half-hour intervals. These are shown by the stepped lines in figures 49 and 50. The curve of instantaneous tidal discharges is then so drawn that the area under the curve during each time interval is the same as the area of the rectangle of mean discharge for the interval.

The fresh-water discharge is plotted as a horizontal line *above* the zero line of tidal discharges. Since the fresh-water discharge is flowing

outward, and therefore intrinsically negative in sign, the difference between the ordinates of the instantaneous tidal and the fresh-water discharge is the total discharge at the instant. These total discharges are then the ordinates measured from the line of the fresh-water discharge.

The area of the cross section at the Ship John at the heights shown by the tide curve is plotted on the diagram for the station. By taking

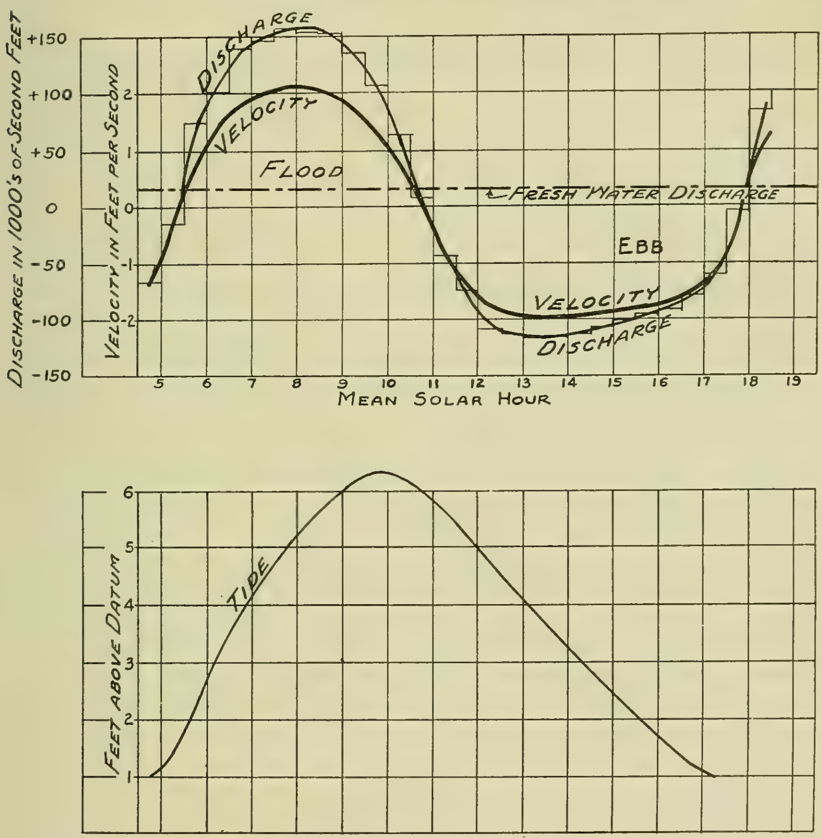

FIGURE 50.—Discharges and velocities in Delaware River at Philadelphia, from mean cubature.

off from the diagram the total discharge and the cross sectional area at a given instant, a refined value of the current velocity at that instant may be computed.

313. An examination of these curves shows that at Ship John Light the average duration of the rise of the tide is about 1 hour less than the duration of the fall; and that the duration of the ebb current is nearly 45 minutes longer than the duration of the flood. At Philadelphia, the average duration of the rise of the tide is about 2 hours less than the average duration of the fall; and the duration of the ebb

exceeds the duration of the flood by more than 2½ hours. A study of the figures shows that the differences in the durations of the flood and ebb currents is to be ascribed principally to the larger areas of the cross sections of the river during the flood; because of which the tidal prism is filled in a shorter period. The fresh-water discharge evidently is insufficient to have any large effect upon the tidal flow except in the upper reaches of the river.

314. Other characteristics of the flow in the estuary, such as the relative timing of the tides and currents, the average and maximum discharges and currents, are quantitatively and definitely brought out by the diagrams at the successive stations. The total volumes of the inflow and outflow at the stations during the tidal cycle are readily derived by measuring the areas between the instantaneous discharges and the line of fresh-water discharge.

315. *Conclusion.*—The cubature of a tidal channel affords complete and reliable data on the discharge and mean velocities at successive stations along the channel. It is perhaps the only means by which a satisfactory determination of the discharge in the wide sections in the lower part of an estuary may be secured. On the other hand the cubature of a long tidal channel is a costly undertaking. It affords no information on the distribution of the velocities in a cross section of the channel, or of the distribution of the flow through the channels on either side of islands and through other secondary channels. Direct measurements of the current velocities in the ship channel of an estuary are of far greater value to navigators than most refined computations of the mean velocities throughout the entire cross section, and are more readily made. The proper design of training works also may depend principally upon the distribution of the velocities in the cross section. For these reasons, extensive cubatures of tidal channels have not often been made. Nevertheless the complete and convincing data afforded by a detailed cubature of the tides in a channel is of such value in the planning of works dependent upon the discharges and velocities that its cost is fully justified when major works of this character are under consideration. Thus the application of the principles of cubature to the estuaries of the Sacramento and San Joaquin Rivers in California afforded information essential to the study of a proposal to construct, at great cost, a barrier dam to prevent the intrusion of salt water into the lower reaches of these rivers. The cubature of the Delaware River has corrected misconceptions of the influence of fresh-water discharge upon its tides and currents, and has contributed to the measures by which the large expense of maintaining the ship channel in the river has been greatly reduced.

In summary, a cubature of an estuary affords much desirable information, but is not warranted unless the information is worth its cost.

Chapter VII

FRICTIONLESS FLOW IN A LONG CANAL OF UNIFORM DIMENSIONS

316. Frictional resistance to flow must, in fact, be considerable in the deepest artificial channel that can be conceived of, if the currents are sufficient to be of any consequence; but the inclusion of frictional resistance imposes insuperable limitations on a general analysis of the flow in a long tidal canal. An analysis of the tides and currents that would be created by frictionless flow in a long canal of uniform cross section, while not affording a quantitive determination of the tides and currents in an actual canal, develops certain general characteristics of the flow in such canals, and affords a background for the procedure, explained in the next chapter, for computing the actual tides and currents. In this analysis of frictionless flow, the currents are considered to be so moderate that the velocity head term of the general equation of motion also may be dropped; and the channel so deep with respect to the tidal range that the variations in the mean depth of the channel, as the tide rises and falls, may be disregarded.

317. The tide at any station in long tidal canal, whether connecting or closed, may be presumed to be the resultant of semidiurnal and diurnal harmonic components of the speeds established in chapter II. The height of the tide above mean sea level at the time t, at a station distant x from the origin of distances, is then:

$$y = \mathrm{M}_2 \cos(\mathrm{m}_2 t + \alpha_1) + \mathrm{S}_2 \cos(\mathrm{s}_2 t + \alpha_2) + \cdot\ \cdot\ \cdot \qquad (190)$$

The amplitudes, M_2, S_2, etc., and the phases α_1, α_2, etc., of the several components may vary with the distance x, of the station from the origin. This origin is conveniently taken at either entrance to a connecting canal, or at the single entrance to a closed canal. The end so chosen will be termed the initial end.

Expanding the cosines in equation (190):

$$y = M_2 \cos \alpha_1 \cos m_2t - M_2 \sin \alpha_1 \sin m_2t + S_2 \cos \alpha_2 \cos s_2t - S_2 \sin \alpha_2 \sin s_2t + \cdots \quad (191)$$

Equation (191) may be written:

$$y = X_1 \cos m_2t + Y_1 \sin m_2t + X_2 \cos s_2t + Y_2 \sin s_2t + \cdots \quad (192)$$

In which X_1, Y_1, X_2, Y_2, etc., are functions of x.

318. When the flow is frictionless, the current is likewise the resultant of component currents having the speeds of the harmonic components of the tide (par. 289).

The velocity at the time t is then given by an equation in the form:

$$v = V_1 \cos m_2t + Z_1 \sin m_2t + V_2 \cos s_2t + Z_2 \sin s_2t + \cdots \quad (193)$$

in which V_1, Z_1, V_2, Z_2, etc., are similarly functions of x.

The form of the functions, X, Y, Z, and V necessary to satisfy the equation of fluid motion for frictionless flow, and the equation of continuity, is then to be determined.

319. When the frictional and velocity head terms are omitted the equation of motion is (equation 176):

$$\partial y/\partial x + (1/g)\partial v/\partial t = 0$$

And in a channel of uniform width and depth, with a tidal fluctuation small in comparison with the depth, the equation of continuity is (equation 184):

$$\partial y/\partial t + D\partial v/\partial x = 0$$

Substituting the differential coefficients derived from equations (192) and (193), the equation of motion becomes:

$$\begin{gathered}(\partial X_1/\partial x)\cos m_2t + (\partial Y_1/\partial x)\sin m_2t + (\partial X_2/\partial x)\cos s_2t \\ + (\partial Y_2/\partial x)\sin s_2t + \cdots - (m_2V_1/g)\sin m_2t + (m_2Z_1/g)\cos m_2t \\ - (s_2V_2/g)\sin s_2t + (s_2Z_2/g)\cos s_2t - \cdots = 0\end{gathered}$$

or:

$$\begin{gathered}(\partial X_1/\partial x + m_2Z_1/g)\cos m_2t + (\partial Y_1/\partial x - m_2V_1/g)\sin m_2t \\ + (\partial X_2/\partial x + s_2Z_2/g)\cos s_2t + (\partial Y_2/\partial x - s_2V_2/g)\sin s_2t + \cdots = 0 \quad (194)\end{gathered}$$

And the equation of continuity becomes:

$$-\mathrm{m}_2X_1 \sin \mathrm{m}_2t+\mathrm{m}_2Y_1\cos \mathrm{m}_2t-\mathrm{s}_2X_2 \sin \mathrm{s}_2t+\mathrm{s}_2Y_2 \cos \mathrm{s}_2t+\cdots$$
$$+D(\partial V_1/\partial x)\cos \mathrm{m}_2t+D(\partial Z_1/\partial x)\sin \mathrm{m}_2t+D(\partial V_2/\partial x)\cos \mathrm{s}_2t$$
$$+\mathrm{D}(\partial Z_2/\partial x)\sin \mathrm{s}_2t \cdots =0$$

or:

$$(D\partial V_1/\partial x+\mathrm{m}_2Y_1) \cos \mathrm{m}_2t+(D\partial Z_1/\partial x-\mathrm{m}_2X_1) \sin \mathrm{m}_2t$$
$$+(D\partial V_2/\partial x+\mathrm{s}_2Y_2)\cos \mathrm{s}_2t+(D\partial Z_2/\partial x-\mathrm{s}_2X_2)\sin \mathrm{s}_2t+\cdots=0 \qquad (195)$$

Equations (194) and (195) are satisfied by all values of t, if:

$$\begin{array}{ll}
\partial X_1/\partial x+\mathrm{m}_2Z_1/g=0 & \partial X_2/\partial x+\mathrm{s}_2Z_2/g=0, \text{ etc.} \\
\partial Y_1/\partial x-\mathrm{m}_2V_1/g=0 & \partial Y_2/\partial x-\mathrm{s}_2V_2/g=0, \text{ etc.} \\
D\partial V_1/\partial x+\mathrm{m}_2Y_1=0 & D\partial V_2/\partial x+\mathrm{s}_2Y_2=0, \text{ etc.} \\
D\partial Z_1/\partial x-\mathrm{m}_2X_1=0 & D\partial Z_2/\partial x-\mathrm{s}_2X_2=0, \text{ etc.}
\end{array}$$

320. *Expressions for the components of the tide.*—An examination of these equations shows that the variable coefficients for each of the tidal and current components are related by the equations:

$$\partial X/\partial x+aZ/g=0 \qquad (196)$$
$$\partial Y/\partial x-aV/g=0 \qquad (197)$$
$$D\partial V/\partial x+aY=0 \qquad (198)$$
$$D\partial Z/\partial x-aX=0 \qquad (199)$$

in which a is the speed of the component.

From equations (196) and (197)

$$Z=-(g/a)\partial X/\partial x \qquad V=(g/a)\partial Y/\partial x \qquad (200)$$

whence:

$$\partial Z/\partial x=-(g/a)\partial^2X/\partial x^2 \qquad \partial V/\partial x=(g/a)\partial^2Y/\partial x^2 \qquad (201)$$

Substituting these expressions in equations (198) and (199):

$$\partial^2Y/\partial x^2+(a^2/gD)Y=0 \qquad (202)$$
$$\partial^2X/\partial x^2+(a^2/gD)X=0 \qquad (203)$$

Evidently the solution of equation (203) will afford also the solution of equation (202).

Placing for convenience, $gD=c^2$, and multiplying the terms in equation (203) by $2\partial X/\partial x$ this equation becomes:

$$2(\partial X/\partial x)(\partial^2X^2/\partial x^2)+2(a^2/c^2)X\partial X/\partial x=0 \qquad (204)$$

The integration of which gives the equation:

$$(\partial X/\partial x)^2+(a^2/c^2)X^2=K^2 \qquad (205)$$

in which K^2 is a constant of integration.

From equation (205):

$$\partial X/\sqrt{K^2-(aX/c)^2}=\partial x \tag{206}$$

The integration of which gives:

$$\sin^{-1}[(aX/c)/K]=ax/c+K', \tag{207}$$

in which K' is a second constant of integration.

From equation (207):

$$X=(cK/a)\sin(ax/c+K') \tag{208}$$

This expression for X may be placed in the form:

$$X=M\cos(ax/c)+N\sin(ax/c) \tag{209}$$

in which M and N are constants.

Since the differential equation (202) for Y is the same as that for X, the expression for Y is similarly:

$$Y=P\cos(ax/c)+Q\sin(ax/c), \tag{210}$$

in which P and Q are constants whose values are independent of those of M and N.

The height of a component of the tide in the canal at a station distant x from the origin of distances, is then given by an equation in the form:

$$\begin{aligned} y=&[M\cos(ax/c)+N\sin(ax/c)]\cos at \\ &+[P\cos(ax/c)+Q\sin(ax/c)]\sin at \end{aligned} \tag{211}$$

321. *Expressions for components of the current.*—The component of the current due to the same component of the tide is, from equation (193):

$$v=V\cos at+Z\sin at \tag{212}$$

From equations (200) and (210):

$$\begin{aligned} V=(g/a)\partial Y/\partial x&=(g/a)[-(a/c)P\sin(ax/c)+(a/c)Q\cos(ax/c)] \\ &=(g/c)[Q\cos(ax/c)-P\sin(ax/c)] \end{aligned} \tag{213}$$

And from equations (200) and (211):

$$\begin{aligned} Z=-(g/a)\partial X/\partial x&=-(g/a)[-(a/c)M\sin(ax/c)+(a/c)N\cos(ax/c)] \\ &=-(g/c)[N\cos(ax/c)-M\sin(ax/c)] \end{aligned} \tag{214}$$

The component of the current at a point distant x from the origin is then:

$$v=(g/c)[Q\cos(ax/c)-P\sin(ax/c)]\cos at$$
$$-(g/c)[N\cos(ax/c)-M\sin(ax/c)]\sin at. \tag{215}$$

The constants M, N, P, and Q in the equation of the current are the same as those in the expression for the corresponding component of the tide.

322. *Determination of the constants.*—The constants in equations (211) and (215) may be determined from the amplitudes and phases of the corresponding component of the tide at the two ends of the canal.

Let L be the length of the canal, A_0 the amplitude and α_0 the phase of the component at the initial end, and A_1 and α_1 the amplitude and phase of this component at the other end.

The equation of the component tide at the initial end is then:

$$y_0=A_0\cos(at+\alpha_0)=A_0\cos at\cos\alpha_0-A_0\sin at\sin\alpha_0 \tag{216}$$

and at the other end:

$$y_1=A_1\cos(at+\alpha_1)=A_1\cos at\cos\alpha_1-A_1\sin at\sin\alpha_1 \tag{217}$$

At the initial end, $x=0$, and equation (211) becomes:

$$y_0=M\cos at+P\sin at \tag{218}$$

Since this equation must be identical with equation (216)

$$M=A_0\cos\alpha_0 \tag{219}$$

$$P=-A_0\sin\alpha_0 \tag{220}$$

At the other end of the canal, $x=L$ and equation (211) becomes:

$$y_1=[M\cos(aL/c)+N\sin(aL/c)]\cos at$$
$$+[P\cos(a\,L/c)+Q\sin(aL/c)]\sin at$$

Therefore:

$$M\cos(aL/c)+N\sin(aL/c)=A_1\cos\alpha_1 \tag{221}$$

$$P\cos(aL/c)+Q\sin(aL/c)=-A_1\sin\alpha_1 \tag{222}$$

It will be convenient to place:

$$aL/c=\gamma \tag{223}$$

It may be noted that γ (gamma) is an angle which is measured in radians, if a is expressed in radians per second, or in degrees, if a is expressed in degrees per second:

Substituting the expressions for M and P, from equations (219) and (220) in equations (221) and (222), and solving for N and Q

$$N = (A_1 \cos \alpha_1 - A_0 \cos \alpha_0 \cos \gamma)/\sin \gamma \tag{224}$$

$$Q = -(A_1 \sin \alpha_1 - A_0 \sin \alpha_0 \cos \gamma)/\sin \gamma \tag{225}$$

323. *Component tides.*—Substituting the expressions for the constants found in the last paragraph, equation (211) becomes:

$$\begin{aligned}
y = {} & [A_0 \cos \alpha_0 \cos (ax/c) \\
& + (A_1 \cos \alpha_1 - A_0 \cos \alpha_0 \cos \gamma) \sin (ax/c)/\sin \gamma] \cos at \\
& - [A_0 \sin a_0 \cos (ax/c) \\
& + [(A_1 \sin \alpha_1 - A_0 \sin \alpha_0 \cos \gamma) \sin (ax/c)/\sin \gamma)] \sin at \\
= {} & \{A_0 \cos \alpha_0 [\sin \gamma \cos (ax/c) - \cos \gamma \sin (ax/c)] \\
& + A_1 \cos \alpha_1 \sin (ax/c)\} \cos at/\sin \gamma \\
& - \{A_0 \sin \alpha_0 [\sin \gamma \cos (ax/c) - \cos \gamma \sin (ax/c)] \\
& + A_1 \sin \alpha_1 \sin (ax/c)\} \sin at/\sin \gamma \\
= {} & [A_0 \cos \alpha_0 \sin (\gamma - ax/c) + A_1 \cos \alpha_1 \sin (ax/c)] \cos at/\sin \gamma \\
& - [A_0 \sin \alpha_0 \sin (\gamma - ax/c) + A_1 \sin \alpha_1 \sin (ax/c)] \sin at/\sin \gamma \\
= {} & [(A_0 \cos \alpha_0 \cos at - A_0 \sin \alpha_0 \sin at) \sin (\gamma - ax/c) \\
& + (A_1 \cos \alpha_1 \cos at - A_1 \sin \alpha_1 \sin at) \sin (ax/c)]/\sin \gamma \\
= {} & A_0 \cos (at + \alpha_0) \sin (\gamma - ax/c)/\sin \gamma \\
& + A_1 \cos (at + \alpha_1) \sin (ax/c)/\sin \gamma
\end{aligned} \tag{226}$$

Since, from equation (223):

$$ax/c = (x/L)\gamma \tag{227}$$

equation (226) also may be written:

$$\begin{aligned}
y = {} & A_0 \cos (at + \alpha_0) \sin (1 - x/L)\gamma/\sin \gamma \\
& + A_1 \cos (at + \alpha_1) \sin (x/L) \gamma/\sin \gamma
\end{aligned} \tag{228}$$

324. *Component currents.*—The equation of the corresponding component of the current, obtained by substituting in equation (215) the same expressions for M, N, P, and Q, similarly reduces to:

$$\begin{aligned}
v = {} & (g/c)A_0 \sin (at + \alpha_0) \cos (\gamma - ax/c)/\sin\gamma \\
& - (g/c)A_1 \sin (at + \alpha_1) \cos (ax/c)/\sin \gamma
\end{aligned} \tag{229}$$

And this equation may be further transformed into:

$$\begin{aligned}
v = {} & (g/c)A_0 \sin (at + \alpha_0) \cos (1 - x/L)\gamma/\sin \gamma \\
& - (g/c)A_1 \sin (at + \alpha_1) \cos (x/L)\gamma/\sin \gamma
\end{aligned} \tag{230}$$

325. *Computation of tides and currents produced by frictionless flow in a connecting canal.*—The tides and currents in a connecting canal are determined by the known amplitudes A_0 and A_1, and initial phases α_0 and α_1, of the several components at the two entrances to the canal. As shown in paragraph 239, equation (228) may be reduced to the form:

$$y = A \cos (at + \alpha)$$

by placing:

$$A \cos \alpha = A_0 \cos \alpha_0 \sin (1 - x/L)\gamma/\sin \gamma + A_1 \cos \alpha_1 \sin (x/L)\gamma/\sin \gamma \quad (231)$$

$$A \sin \alpha = A_0 \sin \alpha_0 \sin (1 - x/L)\ \gamma/\sin \gamma + A_1 \sin \alpha_1 \sin (x/L)\gamma/\sin \gamma \quad (232)$$

The value of γ in degrees, for any component of the tide, is given by equation (223):

$$\gamma = aL/c = aL/\sqrt{gD}$$

in which L is the length of the canal, D its mean depth, and a the speed of the component, in degrees per second. Thus the value of a for the M_2 component is $28°.9841/3600 = 0°.00805$.

The initial phase, α, and the amplitude A of each component of the tide at a point distant x from the origin of distances may be determined from the values of $A \cos \alpha$ and $A \sin \alpha$, equations (231) and (232).

The equation of the a component of the current in the canal at a point distant x from the origin, equation (230), may be reduced to the form:

$$v = B \sin (at + \beta)$$

by a similar procedure.

326. *Instantaneous profiles and wave lengths.*—The longitudinal section of the water surface in a long tidal channel at any instant is a curve designated as the *instantaneous profile.* The instantaneous profile of a component of the tide at any time, t_0, in a long connecting canal of uniform cross section with frictionless flow, is derived at once by placing $t = t_0$ in equation (226).

This equation then takes the form:

$$y = C \sin (ax/c - \gamma) + C' \sin (ax/c) \quad (233)$$

in which

$$C = -A_0 \cos (at_0 + \alpha_0)/\sin \gamma, \text{ and } C' = A_1 \cos (at_0 + \alpha_1)/\sin \gamma$$

This equation is readily transformed into:

$$y = W \sin\,(ax/c + w) \tag{234}$$

The graph of this equation, if sufficiently extended, is the sinusoidal curve shown in figure 51.

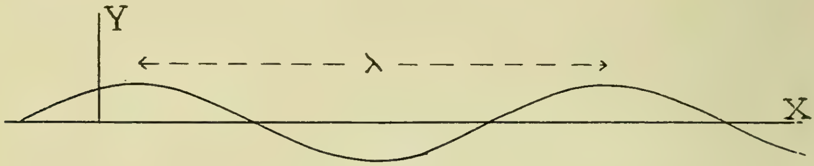

FIGURE 51.—Instantaneous Profile.

The distance, λ (lambda) from crest to crest of the profile is designated its *wave length.* The crests of the sinusoidal curve representing equation (234) are at the points at which:

$$ax_0/c + w = \pi/2, \quad ax_1/c + w = 5\pi/2, \text{ etc.}$$

So that:

$$\lambda = x_1 - x_0 = 5\pi c/2a - \pi c/2a = 2\pi c/a = 2\pi\sqrt{gD}/a \tag{235}$$

In which a is the speed of the component, in radians per second, D the mean depth of the canal, and g the acceleration of gravity. The wave length of the M_2 component of the tide, in a channel whose mean depth is 30 feet, is, for example

$$\lambda = \frac{2\pi\sqrt{30g}}{0.00014053} = 1{,}389{,}000 \text{ feet} = 263 \text{ miles}$$

Evidently, the length of a canal is usually but a small fraction of the wave length of its principal tidal components.

The form of equation (229) shows that the graph of the instantaneous component velocities through a long canal of uniform dimensions with frictionless flow is a similar sinusoidal curve with the same wave length as the instantaneous profile.

327. *Relation between* γ *and* λ.—From equations (223) and (235)

$$\gamma = aL/c = 2\pi L/\lambda \tag{236}$$

It may be noted that if the length, L, of a connecting canal is one-half the wave length, λ, of a component of the tide, $\gamma = \pi$ and $\sin \gamma = 0$. As subsequently discussed in paragraph 347, the tides and currents in a canal of this length would become infinite if there were no frictional resistance to flow.

328. *Example of frictionless tides and currents in a long connecting canal.*—The characteristics of frictionless flow in a connecting canal whose length is less than one-half of the wave length of the tidal components, may be exemplified by the tides and currents that would be produced in a canal 200,000 feet (37.8 miles) in length, of uniform width, and with a mean depth of 30 feet, by the M_2 component of the tide, if its range at the initial entrance is 8 feet, and at the other entrance 4 feet, high water at the latter being 2 lunar hours, or 60°, earlier than at the former. Taking the origin of time at an instant of high water at the initial entrance, the given data are:

$$A_0=4 \text{ feet}; \alpha_0=0; A_1=2 \text{ feet}; \alpha_1=60°;$$
$$L=200{,}000 \text{ feet}; D=30 \text{ feet}; c=\sqrt{gD}=31.06;$$
$$a=\mathrm{m}_2=0°.00805 \text{ per second}; \gamma=\mathrm{m}_2L/c=51°.83=51°50'.$$

The computation of the amplitudes and phases of the tide and current at the entrances to the canal ($x=0$ and $x=L$) and at its quarter and mid points ($x=\frac{1}{4}L$, $\frac{1}{2}L$, and $\frac{3}{4}L$) is summarized in the following tabulation:

x	0	$\frac{1}{4}L$	$\frac{1}{2}L$	$\frac{3}{4}L$	L
$(1-x/L)\gamma$	51°50′	38°52′.5	25°55′	12°57′.5	0
$(x/L)\gamma$	0	12°57′.5	25°55′	38°52′.5	51°50′
$A \sin \alpha$	0	0.494	0.963	1.383	1.732
$A \cos \alpha$	4	3.478	2.780	1.939	1.000
α	0	8°05′	19°06′	35°30′	60°
A	4	3.513	2.942	2.382	2.000
$B \sin \beta$	−2.281	−2.223	−2.052	−1.776	−1.410
$B \cos \beta$	1.938	2.818	3.554	4.109	4.454
β	−49°39′	−38°15′	−30°0′	−23°22′	−17°34′
B	2.993	3.588	4.104	4.476	4.672

The equations of the tides and currents then are:
At the initial end:

$$y=4 \cos \mathrm{m}_2t$$
$$v=2.99 \sin (\mathrm{m}_2t-49°39')$$

At the first quarter point:

$$y=3.51 \cos (\mathrm{m}_2t+8°5')$$
$$v=3.59 \sin (\mathrm{m}_2t-38°15')$$

At the middle:

$$y=2.94 \cos (\mathrm{m}_2t+19°6')$$
$$v=4.10 \sin (\mathrm{m}_2t-30°)$$

At the third quarter point:

$$y=2.38 \cos (\mathrm{m}_2t+35°30')$$
$$v=4.48 \sin (\mathrm{m}_2t-23°22')$$

At the further end:

$$y=2 \cos (m_2t+60°)$$
$$v=4.67 \sin (m_2t-17°34')$$

Since the speed of any competent, in degrees per component hour (par. 84) is $360°/12=30°$, and the component hour of the M_2 component is the mean lunar hour of 1.035 mean solar hours, the tides and currents represented by these equations are most conveniently plotted in terms of lunar hours, by placing $m_2t=0, 30°, 60°$, etc. The tides and currents at the entrances and at the middle of the canal, and the instantaneous profiles at the successive lunar hours, designated as 0, I, II, III, etc., are plotted in figure 52.

It will be noted that the amplitudes and phases of the tides and currents go through a progressive, but not uniform, variation from one end of the canal to the other, and that the amplitude of the current increases as that of the tide decreases.

329. *Progression of high and low waters, and of the strength and turn of the current, through a connecting canal.*—The times of high and low water, of the strengths of the positive and negative currents, and of the turn of the current, at points along the canal may be determined immediately from the phases of the tides and currents at these points. Thus in the example set forth in paragraph 328, in which the origin of time was taken at a high water at the initial entrance to the canal, the current at this entrance turns when $v=0$ and $m_2t-49°.6=0$. As the speed of the component is 28°.98 per solar hour, the turn of the current occurs $49°.6/28°.98=1.71$ solar hours *after* high water; and the strength of the positive current $90°/28°.98=3.11$ hours after the turn, or 4.82 hours after high water. Similarly, at the first quarter point the phase of the tide is 8°.1, and high water occurs when $t=-8°.1/28°.98$, or 17 minutes *before* high water at the entrance; so that if high water at the entrance is at noon, high water at the quarter point is at 11.43 a. m. A plot of these times at successive points along the canal (fig. 53, page 168) shows how high and low water and the strength of and turn of the current, progress through it.

Obviously, high water and low water must travel through a connecting canal at such rates as to reach the far end at the time of high and low waters at that entrance, the total time of travel being fixed by the difference in the timing of the tides at the two entrances; but the rate of travel is not in general uniform. The progress of the strength and turn of the current through the canal is determined, on the other hand, by the rate of storage and release of water in the successive sections of the canal, and is dependent on the depth of the canal. No fixed relation exists therefore in the general case between the progress of high water and of the strength of the current through a connecting canal.

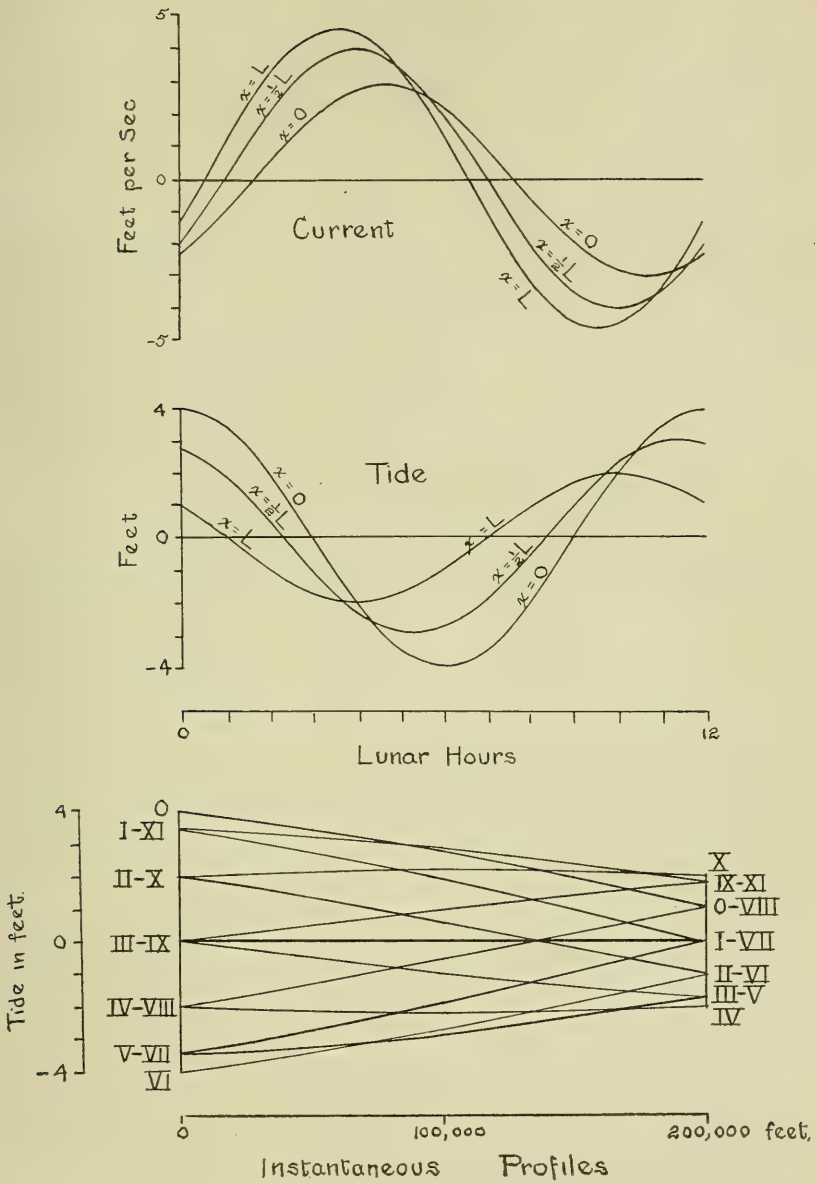

FIGURE 52.—Frictionless tides and currents in connecting canal.

330. *Tides and currents at the middle of a connecting canal.*—At the middle of a canal, $x=\frac{1}{2}L$ and equation (228) becomes:

$$y_m=A_0 \cos (at+\alpha_0) \sin \tfrac{1}{2}\gamma/\sin \gamma+A_1 \cos (at+\alpha_1) \sin \tfrac{1}{2}\gamma/\sin \gamma$$

Since $\sin \gamma=2 \sin \frac{1}{2} \gamma \cos \frac{1}{2} \gamma$ this equation reduces to:

$$y_m=\tfrac{1}{2}[A_0 \cos (at+\alpha_0)+A_1 \cos (at+\alpha_1)]/\cos \tfrac{1}{2}\gamma \qquad (237)$$

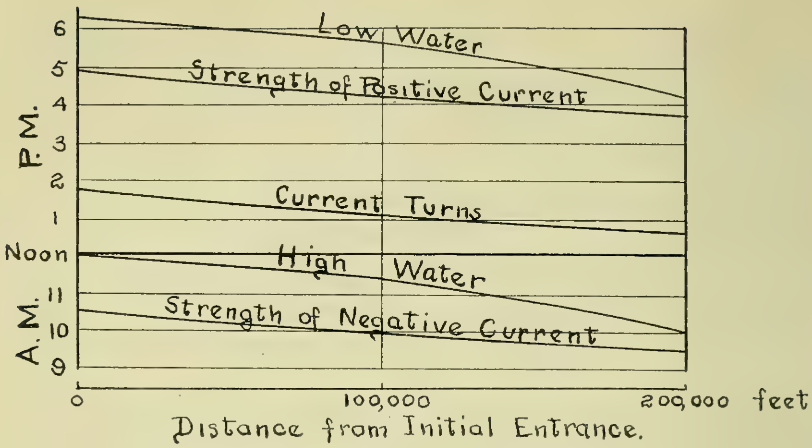

FIGURE 53.—Progression of tide and current through connecting canal.

The height of the tide at the middle of the canal is then the mean of the heights at the entrances, increased by the factor $1/\cos \frac{1}{2}\gamma$.

The velocity at the middle of the canal is, similarly, from equation (230):

$$\begin{aligned} v_m &= (g/c)[A_0 \sin (at+\alpha_0) - A_1 \sin (at+\alpha_1)] \cos \tfrac{1}{2}\gamma / \sin \gamma \\ &= \tfrac{1}{2}(g/c)[A_0 \sin (at+\alpha_0) - A_1 \sin (at+\alpha_1)]/\sin \tfrac{1}{2}\gamma \end{aligned} \tag{238}$$

The total surface head between the two entrances to the canal is:

$$h_s = A_1 \cos (at+\alpha_1) - A_0 \cos (at+\alpha_0)$$

If the effect of the tidal storage were neglected, the water surface would have the uniform slope of:

$$\partial y/\partial x = [A_1 \cos (at+\alpha_1) - A_0 \cos (at+\alpha_0)]/L \tag{239}$$

And from equation (177) the velocity through the canal would be:

$$v_1 = -g\int (\partial y/\partial x)\,\partial t = (g/aL)[A_0 \sin (at+\alpha_0) - A_1 \sin (at+\alpha_1)] \tag{240}$$

The ratio of the velocity at the middle of the canal, when tidal storage is considered, to the velocity through the canal without tidal storage, is then, from equations (238) and (240):

$$v_m/v_1 = \frac{\tfrac{1}{2}(g/c)/\sin \tfrac{1}{2}\gamma}{g/aL} = (aL/c)/2 \sin \tfrac{1}{2}\gamma = \gamma/2 \sin \tfrac{1}{2}\gamma \tag{241}$$

In equation (241), γ is measured in radians. Its relation to the wave length, λ, of the component is given by equation (236):

$$\gamma = 2\pi L/\lambda = L/(\lambda/2\pi)$$

331. The nature of the ratios, $1/\cos \frac{1}{2}\gamma$ and $\gamma/2 \sin \frac{1}{2}\gamma$ can perhaps be shown more clearly on a diagram.

Let $A\ B\ C\ F$ figure 54, be a circle of radius $\lambda/2\pi$, and hence with a circumference of length equal to λ; and let $A\ B\ C$ be an arc of length L. The subtended central angle AOC is then $L/(\lambda/2\pi)=\gamma$ radians, and the length of the chord AC is $2(\lambda/2\pi)$. $\sin \frac{1}{2}\gamma$. The ratio v_m/v_1 is therefore the ratio of the arc ABC to the chord AC; and $1/\cos \frac{1}{2}\gamma$ is the ratio of OC to OD. It is apparent therefore, that if the length of a canal is not too large a part of the wave length of the tidal components, frictionless tides and currents at its middle do not differ greatly from the values which they would have if the effect of tidal storage were neglected.

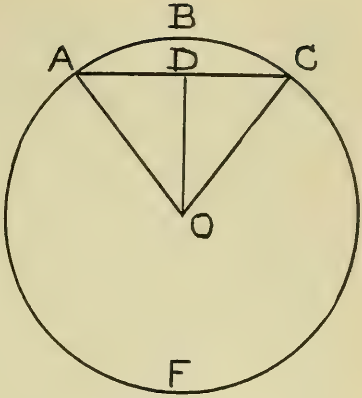

FIGURE 54.

SPECIAL CASES

332. *A canal connecting a tidal with a tideless sea.*—If the sea at one end of the canal is tideless, all of the components there have a zero amplitude and equations (228) and (230) become:

$$y=A_0 \cos (at+\alpha_0) \sin (1-x/L)\gamma/\sin \gamma \qquad (242)$$

$$v=(g/c)A_0 \sin (at+\alpha_0) \cos (1-x/L)\gamma/\sin \gamma \qquad (243)$$

It is apparent that, when the equations reduce to this form, y becomes a maximum and v becomes zero, for all values of x, when $at+\alpha_0=0$, or when $t=-\alpha_0/a$; and that $y=0$ and v is a maximum, when $t=-\alpha_0/a+\pi/a$. Both high water and low water occur therefore at the same respective instants throughout the canal, and the current turns at the same instants.

If, for example, the canal described in paragraph 328 entered a tideless sea at its further end, the tides and currents at the entrances and at the middle of the canal, and the instantaneous profiles at successive lunar hours, would take the forms shown in figure 55, page 170, were the flow frictionless.

333. *When high water occurs at the same time at both entrances, or when the tides at these entrances are exactly opposite.*—If the phases of the tides at both entrances are the same, $\alpha_1=\alpha_0$, and equations (228) and (230) reduce to:

$$y=\cos (at+\alpha_0)[A_0 \sin (1-x/L)\gamma+A_1 \sin (x/L)\gamma]/\sin \gamma \qquad (244)$$

$$v=(g/c) \sin (at+\alpha_0)[A_0 \cos (1-x/L)\gamma-A_1 \cos (x/L)\gamma]/\sin \gamma \qquad (245)$$

In this case, also, high water and low water each occur at the same instants throughout the canal, and the currents turn throughout the canal at these instants. It is readily shown that the same conditions result if the phases of the tides at the ends of the canal differ by 180°.

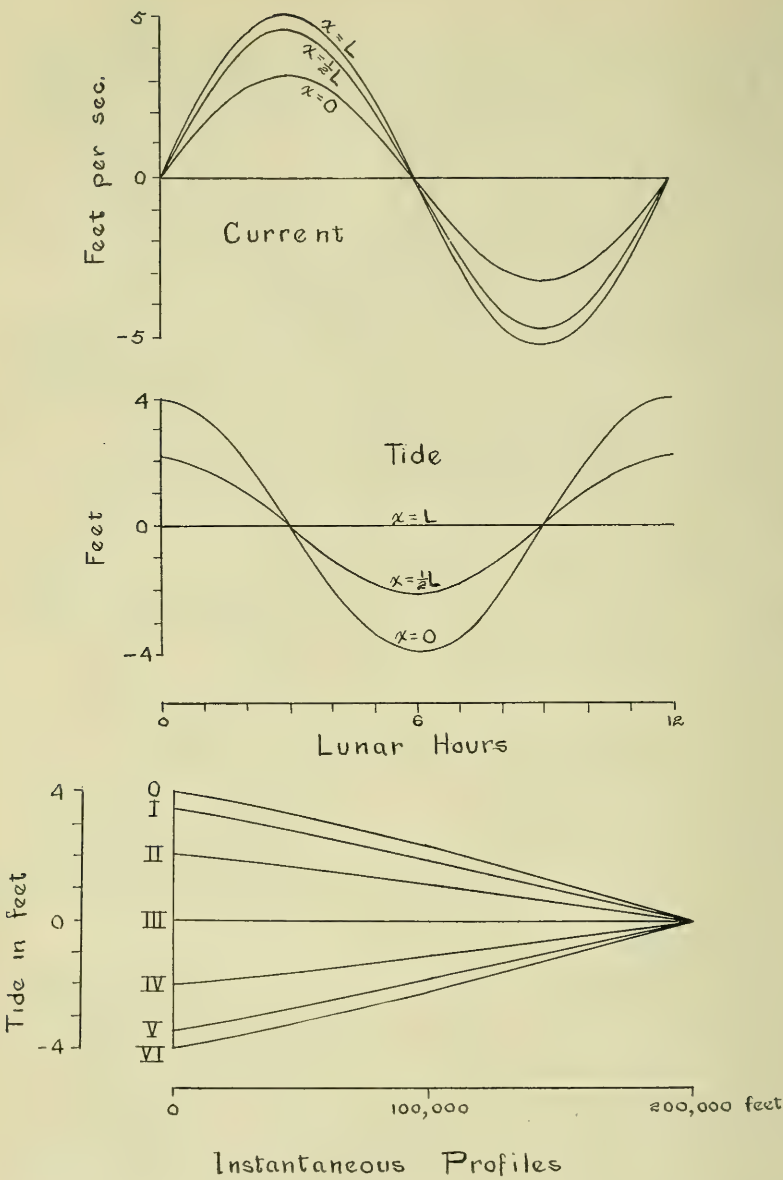

FIGURE 55.—Frictionless tides and currents in canal connecting a tidal with a tideless sea.

334. *Meeting the tides.*—If the tides at the two entrances are identical in range and in timing, equations (244) and (245) further reduce to:

$$y = A_0 \cos (at + \alpha_0)[\sin (1 - x/L)\gamma + \sin (x/L)\gamma]/\sin \gamma \qquad (246)$$

$$v = (g/c)A_0 \sin (at + \alpha_0)[\cos (1 - x/L)\gamma - \cos (x/L)\gamma]/\sin \gamma \qquad (247)$$

and at the middle of the canal:

$$v = (g/c)A_0 \sin (at + \alpha_0)(\cos \tfrac{1}{2}\gamma - \cos \tfrac{1}{2}\gamma)/\sin \gamma = 0$$

Obviously, under such conditions, the total surface head through the canal is always zero, and the currents are due solely to the filling and emptying of the canal from the two entrances. At the middle of the canal the currents disappear, and the tides are said to meet.

FRICTIONLESS TIDES AND CURRENTS IN A CLOSED CANAL

335. The amplitude and initial phase of each component of the tide at the head of a closed canal is determined by the condition that the currents are there zero. Taking the open end of the canal as the initial entrance, the equation of a component of the tide at this entrance is $y_0 = A_0 \cos (at + \alpha_0)$ and at the other end, at the head of the canal, $y_1 = A_1 \cos (at + \alpha_1)$. At the head of the canal $x = L$, the length of the canal, and the velocity of the corresponding component of the current is, from equation (230):

$$v = (g/c) A_0 \sin (at + \alpha_0)/\sin \gamma - (g/c) A_1 \sin (at + \alpha_1) \cos \gamma/\sin \gamma = 0$$

Whence:

$$A_1 \sin (at + \alpha_1) = A_0 \sin (at + \alpha_0)/\cos \gamma \qquad (248)$$

Since this equation is identically true for all values of t:

$$A_1 = A_0/\cos \gamma \qquad (249)$$

$$\alpha_1 = \alpha_0 \qquad (250)$$

Substituting these equivalents in equation (228), the equation of a component of the tide at any point in a closed canal becomes:

$$\begin{aligned}
y &= A_0 \cos (at + \alpha_0) \sin (1 - x/L)\gamma/\sin \gamma \\
&\qquad + A_0 \cos (at + \alpha_0) \sin (x/L)\gamma/\sin\gamma \cos \gamma \\
&= A_0 \cos (at + \alpha_0) [(\sin \gamma \cos (x/L)\gamma - \cos \gamma \sin (x/L)\gamma) \cos \gamma \\
&\qquad + \sin (x/L)\gamma]/\sin \gamma \cos \gamma \\
&= A_0 \cos (at + \alpha_0) [\sin \gamma \cos (x/L)\gamma \cos \gamma \\
&\qquad + (1 - \cos^2 \gamma) \sin (x/L)\gamma]/\sin \gamma \cos \gamma \\
&= A_0 \cos (at + \alpha_0) [\cos (x/L)\gamma \cos \gamma + \sin (x/L)\gamma \sin \gamma]/\cos \gamma \\
&= A_0 \cos (at + \alpha_0) \cos (1 - x/L)\gamma/\cos \gamma. \qquad (251)
\end{aligned}$$

The substitution of the same values of A_1 and α_1 in equation (230) gives the equation of the corresponding component of the current, which similarly reduces to:

$$v = -(g/c) A_0 \sin (at + \alpha_0) \sin (1 - x/L)\gamma/\cos \gamma \qquad (252)$$

336. The form of equations (251) and (252) shows that, if the flow were frictionless, high water and low water each would occur simultaneously throughout a closed canal, and the current would turn at these instants at every point in the canal. The maximum currents

would occur at mean tide. The tides and currents at the entrance, the middle, and the end of a closed canal 30 feet in mean depth and 200,000 feet in length, which would be produced by a simple harmonic fluctuation of the tide at the entrance with the speed of the M_2 component, and a range of 8 feet, were the flow frictionless, are shown in figure 56, with the instantaneous profiles through the canal.

337. *Relation of amplitudes of the components of the current to those of the components of the tide in a closed canal.*—From equation (251), the amplitude of a component of the tide at a point distant x from the entrance is:

$$A = A_0 \cos (1 - x/L)\gamma / \cos \gamma \tag{253}$$

and from equation (252) the amplitude of the corresponding component of the current is:

$$B = (g/c) A_0 \sin (1 - x/L)\gamma / \cos \gamma \tag{254}$$

so that:

$$B = (g/c) A \tan (1 - x/L)\gamma \tag{255}$$

Designating the amplitude of any other component of the current at the same point as B_1, the amplitude of the corresponding component of the tide as A_1, and the value of γ for this component as γ_1, then:

$$B_1 = (g/c) A_1 \tan (1 - x/L)\gamma_1$$

and:

$$B_1/B = (A_1/A) \tan (1 - x/L)\gamma_1 / \tan (1 - x/L)\gamma \tag{256}$$

If the length of the canal is a small part of the wave lengths of the components, the angles $(1 - x/L)\gamma$ and $(1 - x/L)\gamma_1$ are small, and are approximately proportional to their tangents.

Then, approximately:

$$B_1/B = (A_1/A)(\gamma_1/\gamma) \tag{257}$$

Since $\gamma = aL/c$ and $\gamma_1 = a_1 L/c$, equation (257) becomes:

$$B_1/B = A_1 a_1 / A a \tag{258}$$

It follows, therefore, that unless a closed canal is quite long, the amplitudes of the components of the current produced at a given point by frictionless flow, are nearly proportional to the products of the amplitudes and speeds of the corresponding components of the tide at that point. These speeds, it may be observed, may be expressed in any units.

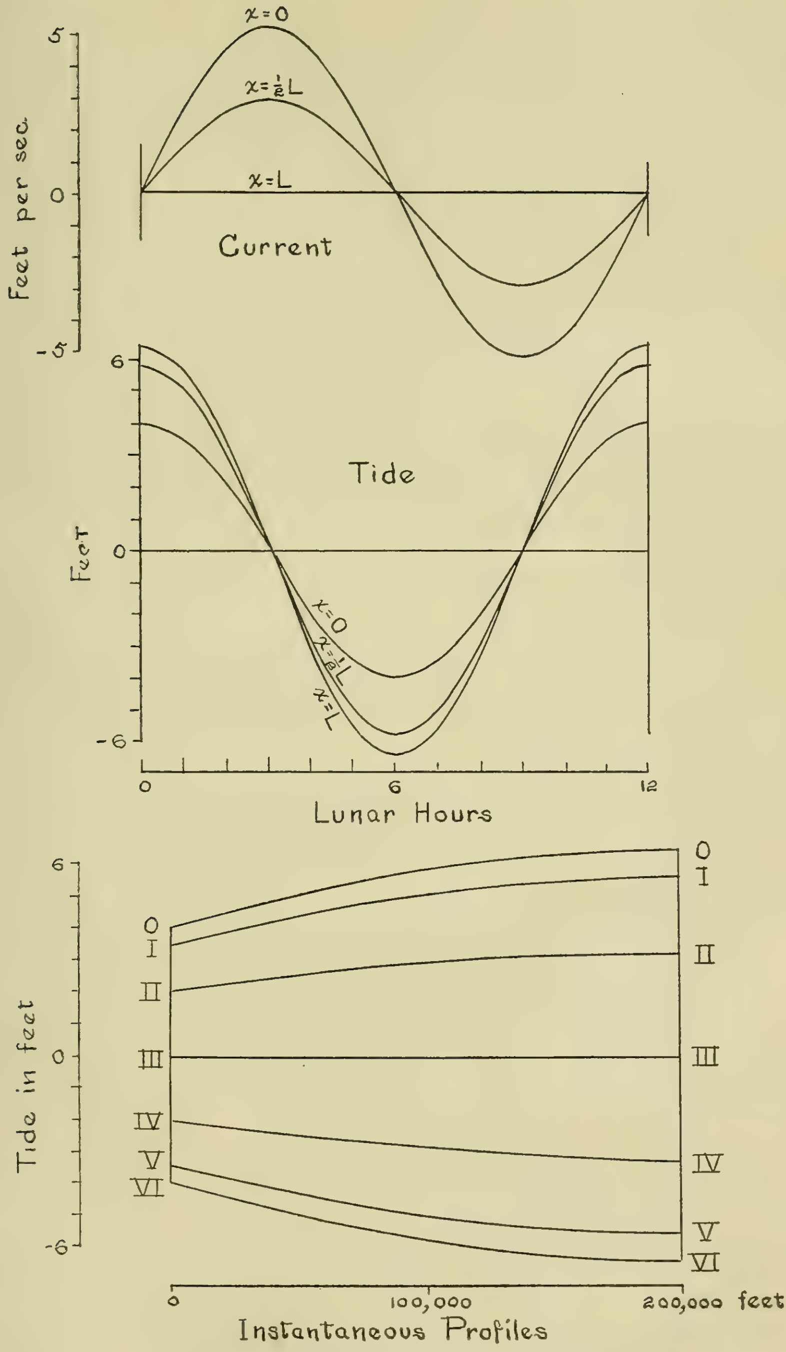

FIGURE 56.—Frictionless tides and currents in closed canal.

The diurnal components of the current in a closed canal have therefor a much smaller ratio to the semidiurnal current components than the diurnal components of the tide have to the semidiurnal tidal components. It is easy to see, indeed, that the filling and emptying of the tidal prism of the canal by the diurnal components is at but half the rate of the filling and emptying by the semidiurnal components. In a connecting canal, on the contrary, the ratio of the diurnal to the semidiurnal components of the current may be increased because of the proportionally large acceleration heads set up by the semidiurnal components.

PROGRESSIVE, RETROGRESSIVE, AND STATIONARY WAVES

338. *The progressive wave.*—A special condition of frictionless tidal flow arises when the amplitude of a component of the tide is the same at both entrances of a connecting canal, and the phases of the component at the two entrances differ by the angle γ. Taking first the case in which high water at the further entrance is later than at the initial entrance, and placing in equation (226), $A_1=A_0$, and $\alpha_1=\alpha_0-\gamma$, this equation becomes:

$$\begin{aligned} y=&A_0 \cos (at+\alpha_0) \sin (\gamma-ax/c)/\sin \gamma \\ &+A_0 \cos (at+\alpha_0-\gamma) \sin (ax/c)/\sin \gamma \end{aligned}$$

Expanding by the formula:

$$\cos A \sin B=\tfrac{1}{2} \sin (A+B)-\tfrac{1}{2} \sin (A-B) \tag{259}$$

$$\begin{aligned} y=&\tfrac{1}{2}A_0[\sin (at+\alpha_0+\gamma-ax/c)-\sin (at+\alpha_0-\gamma+ax/c) \\ &+\sin (at+\alpha_0-\gamma+ax/c)-\sin (at+\alpha_0-\gamma-ax/c)]/\sin \gamma \\ =&\tfrac{1}{2}A_0[\sin (at+\alpha_0-ax/c+\gamma)-\sin (at+\alpha_0-ax/c-\gamma)]/\sin \gamma \\ =&A_0 \cos (at+\alpha_0-ax/c) \sin \gamma/\sin \gamma \qquad (260) \\ =&A_0 \cos (at-ax/c+\alpha_0) \end{aligned}$$

The expression for the current is most readily derived by applying equation (177):

$$v=-g\int (\partial y/\partial x)\partial t$$

From equation (260):

$$\partial y/\partial x=(A_0a/c) \sin (at-ax/c+\alpha_0)$$

Whence:

$$v=-g\int (A_0a/c) \sin (at-ax/c+\alpha_0)\ \partial t=(g/c)A_0 \cos (at-ax/c+\alpha_0) \tag{261}$$

339. From equation (260) it is seen that the component tide has, under these conditions, a constant amplitude throughout the canal. At a point distant x from the origin its high water occurs when $at-ax/c+\alpha_0=0$; or when $t=x/c-\alpha_0/a$.

The time of high water therefore increases with the distance x at the uniform rate of $c=\sqrt{gD}$ feet per second. Successive instantaneous profiles when sufficiently extended, have the relation shown in figure 57.

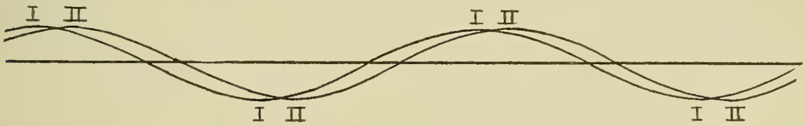

FIGURE 57.—Successive instantaneous profiles of progressive wave.

Evidently, in this case, a wave *progresses* through the canal with a speed, $\sqrt{gD}$, which depends only on the depth, D, of the canal and is independent of the speed of the tidal fluctuations and of the wave length of the component, and independent also of the maximum currents in the canal.

The current velocities (equation 261) similarly have the constant amplitude, $A_0g/c=A_0\sqrt{g/D}$ throughout the canal. At every station along the canal the strength of the current occurs at high water.

340. *Example of a frictionless progressive wave.*—In a canal 200,000 feet in length, with a mean depth of 30 feet, the value of γ for the M_2 component of the tide has been found to be 51°50′. The tidal flow through a canal of these dimensions will then have the form of a progressive wave if the tides at the entrances have a simple harmonic fluctuation with the speed of the M_2 component, the same amplitude, and the phase of the tide at the initial entrances is 51°50′ larger than that at the other entrance. The tide at the farther entrance is then 51°.83/28°.98=1.79 solar hours, or 51°.83/30°=1.73 lunar hours later than at the initial end. If tidal range at the entrances is 8 feet, the equations of the tides and currents in the canal are, when the origin of time is at the time of high water at the initial entrance:

$$y=4\cos(\mathrm{m}_2t-0^\circ.000259x)$$

$$v=4.14\cos(\mathrm{m}_2t-0^\circ.000259x)$$

The currents and tides at the entrances and at the midpoint of the canal, and the instantaneous profiles at successive lunar hours, are shown in figure 58, page 176.

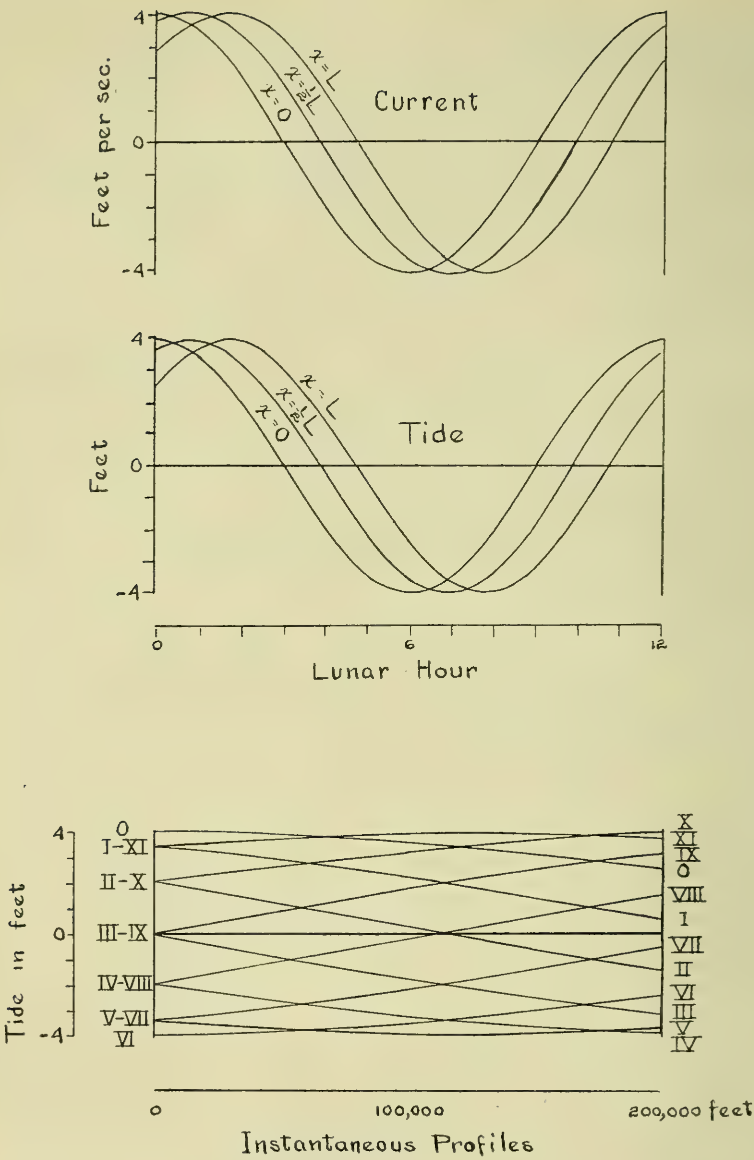

FIGURE 58.—Tides and currents of frictionless progressive wave.

341. It readily may be shown that a progressive wave would be propagated through an *endless* deep canal of uniform section, from any fluctuation of the water surface at the entrace, whether harmonic or otherwise, if the flow were frictionless. Such a wave would also be propagated, under the same conditions, through a canal of finite length, if the energy transmitted could be wholly absorbed at the far end. As will be shown later, the tidal flow in an estuary of the usual

shape, in which the cross section diminishes at such a rate as to counterbalance the friction losses, also takes the form of a progressive wave, but with the current out of phase with the tide. The progressive wave is commonly regarded, therefore, as a normal form of tidal motion in a long channel; but tidal flow does not take the form of a progressive wave in all cases.

342. *The retrogressive wave.*—If the amplitude of a tidal component is the same at both ends of the canal, and its phase at the far end exceeds by the angle γ the phase at the initial end, so that high water is later at the initial end, equations (260) and (261) become:

$$y=A_0 \cos (at+ax/c+\alpha_0) \tag{262}$$

$$v=-(g/c)A_0 \cos (at+ax/c+\alpha_0) \tag{263}$$

A wave then retrogresses through the canal at the rate of $\sqrt{gD}$ feet per second.

343. *Resolution of frictionless tides in a connecting canal into progressive and retrogressive waves.*—The application of equation (259) to the equation of any component of the tide in the form derived in equation (226):

$$y=A_0 \cos (at+\alpha_0) \sin (\gamma-ax/c)/\sin \gamma+A_1 \cos (at+\alpha_1) \sin (ax/c)/\sin \gamma$$

gives:

$$y=\tfrac{1}{2}A_0 \sin (at+\alpha_0+\gamma-ax/c)/\sin \gamma-\tfrac{1}{2}A_0 \sin (at+\alpha_0-\gamma+ax/c)/\sin \gamma$$
$$+\tfrac{1}{2}A_1 \sin (at+\alpha_1+ax/c)/\sin \gamma-\tfrac{1}{2}A_1 \sin (at+\alpha_1-ax/c)/\sin \gamma$$

The first and fourth terms of this equation may be combined into a term in the form:

$$W_1 \cos (at-ax/c+w_1)$$

and the second and third into one in the form:

$$W_2 \cos (at+ax/c+w_2)$$

giving:

$$y=W_1 \cos (at-ax/c+w_1)+W_2 \cos (at+at/c+w_2). \tag{264}$$

From the form of equation (264) it is seen that if the flow were frictionless, the water surface in a connecting canal produced by a component of the tide at the entrances would be the resultant of two waves, one progressive and the other retrogressive. Since the speed, $\sqrt{gD}$, of the waves produced by each component of the tide depends only on the depth in the channel, the resultant of all of the components of the tide is similarly resolvable into two compound waves traveling in opposite directions through the canal. The combination of these two component waves produces, in general, a wave which

travels through the canal with changing amplitude and varying speed. The currents may be correspondingly resolved.

344. The resolution of the frictionless tides in a connecting canal 200,000 feet in length and 30 feet in mean depth, into component waves, when the equation of the tide at the initial and farther entrances are respectively:

$$y = 4 \cos m_2 t \qquad y = 2 \cos (m_2 t + 60°)$$

gives the equation:

$$y = 1.30 \cos (m_2 t - m_2 x/c - 46°10') + 3.24 \cos (m_2 t + m_2 x/c + 16°47')$$

It may be seen, therefore, that the amplitudes and phases of the progressive and retrogressive *waves* in a long connecting canal generally differ widely from the amplitudes and phases of the tides at the entrances, and have no simple relation thereto.

345. *The stationary wave.*—In a closed canal, and in certain special cases in a connecting canal, as has been seen, the phase of the tide and of the current produced by frictionless flow is the same at all points, so that high and low water and the strength and turn of the current, each occurs simultaneously throughout the canal. When the tide in a closed canal, derived in equation (251), is resolved into progressive and retrogressive waves by the process indicated in paragraph 343, its equation becomes:

$$\begin{aligned} y = {} & \tfrac{1}{2}A_0 \cos (at - ax/c + \alpha_0 + \gamma)/\cos \gamma \\ & + \tfrac{1}{2}A_0 \cos (at + ax/c + \alpha_0 - \gamma)/\cos \gamma \end{aligned} \tag{265}$$

These two component waves have the same amplitude, $A_0/2 \cos \gamma$. The retrogressive wave may be regarded as the reflection of the progressive wave from the end of the canal. The resultant of the progressive and retrogressive (or reflected) waves of the same amplitude is a wave which neither advances or retreats, but remains stationary. It will be noted that when a stationary wave is produced by frictionless flow, the current turns at high and low water; while if a simple progressive wave is produced, the strength of the current at each station along the canal occurs at high and low water, and the current turns at midtide.

Frictional resistance must modify the conditions of flow in a long closed canal, since the lag of the current increases as the currents decrease toward the head of the canal. From another viewpoint, the absorption of energy by friction reduces the amplitude of the reflected wave. The tides at the head of a closed channel are therefore always later than those at the entrance, and a completely stationary wave is never found. It is nearly realized in such deep channels as the fiords of Alaska. Thus in the Portland Canal, a fiord from 600 to 1,000 feet

in depth, and of quite regular section, the published high water intervals show that high water at Eagle, at the head of the fiord, is but 2 minutes later than at Halibut Bay, 55 miles down the channel and not far from the entrance. In closed channels of more usual depths, the frictional resistance is considerable if the length of the channel is sufficient to set up any appreciable currents, and the tidal fluctuations may more nearly resemble a wave of the progressive type.

CRITICAL LENGTHS OF A CANAL WERE THE FLOW FRICTIONLESS; NODES IN A CLOSED CANAL

346. *Critical lengths of a closed canal.*—The formulas for the tides and currents which would be produced in a closed canal by a component of the tide at the entrance, were the flow frictionless (equations (251) and (252)) shows that these would reach infinite proportions if the length of the canal were such that $\gamma = \pi/2$, $3\pi/2$, $5\pi/2$, etc., since $\cos \gamma$ would then become zero. Since $\gamma = 2\pi\, L/\lambda$ (equation 236) these critical lengths occur when $L = \lambda/4$, $3\lambda/4$, $5\lambda/4$, etc.; i. e., when the length, L, of the canal is one-quarter, three-quarters, etc., of the wave length, λ, of the component.

It is apparent that if a closed canal is not of great length, the currents set up by the filling and emptying of the tidal prism are moderate, and the slopes produced by a small increase in the tidal ranges in the canal are sufficient to check the momentum of the moving water. As the length of a canal increases, the increase in the tidal ranges in the canal further accentuates the currents, and if these were not restrained by frictional resistance, they would reach infinite proportions if the canal had the critical length of one quarter of the wave length of the component. If the length of the canal exceeds this critical length, the currents at the head of the canal are in the opposite direction to those at the entrance, and the momentum of the water is correspondingly controlled. The currents then are finite until the length of the canal reaches the second critical length; and so on.

347. *Critical lengths of a connecting canal.*—Unless the amplitudes and timing of the component of the tide at the entrances to a connecting canal are such as to produce a simple progressive (or retrogressive) wave, a critical length for the component is reached when $\sin \gamma = 0$, and hence when $L = \frac{1}{2}\lambda$, λ, $1\frac{1}{2}\lambda$, etc. The condition of flow in each half of the canal at the first of these critical lengths is like that in a closed canal of one quarter of the wave length of the component. The water entering through both entrances would pile up in the canal without limit, were there no frictional resistance. When the length of the canal is equal to the wave length of the component, the positive and negative currents exactly balance each other, and the net work done in the acceleration and deceleration of the current becomes zero, so that frictional resistance would alone limit the flow.

Obviously, frictional resistance must control the currents when a connecting or a closed canal approaches its critical lengths.

348. *Nodes in a closed canal.*—The equation of a component of the tide at a point in a closed canal distant x from the entrance is, equation (251):

$$y = A_0 \cos (at + \alpha_0) \cos (1 - x/L)\gamma / \cos \gamma$$

The tide, y, becomes zero, for all values of t, at the points at which $(1-x/L)\gamma = \pi/2$, $3\pi/2$, $5\pi/2$, etc. At these points $x = L - \pi L/2\gamma$, $L - 3\pi L/2\gamma$, $L - 5\pi L/2\gamma$, etc., and hence $x = L - \lambda/4$, $L - 3\lambda/4$, $L - 5\lambda/4$, etc.

If the flow were frictionless and the canal long enough, each component of the tide would then disappear at points one-quarter, three-quarters, etc., of its wave length from the head of the canal. At these points the component current would reach a maximum amplitude of $(g/c)A_0/\cos\gamma$. These points are termed *nodal points.*

Similarly a component current in a closed canal would become zero at the points at which $\sin (1 - x/L)\gamma = 0$, and hence at which:

$$x = L,\ 1 - \tfrac{1}{2}\lambda,\ L - \lambda, \text{ etc.}$$

And at these points the component of the tide would have a maximum amplitude of $A_0/\cos \gamma$.

It is perhaps needless to point out that true nodal points have no counterpart in actual channels.

349. *Shallow water components of frictionless tides and currents.*—The variation of the mean depth, D, in the equation of continuity (equation 184) with the rise and fall of the tide has been neglected in the preceding analysis of the tides and currents in a canal without frictional resistance. This variation must in fact produce distortions of the simple harmonic fluctuations of the components of the tides and currents that have been deduced. It may be shown that these distortions, like those due to the form of the friction term, may be reproduced by overcurrents and compound currents, and overtides and compound tides. As illustrated in the cubature of the Delaware River, in chapter VI, this variation in the depth may produce marked distortions of the current in a long and comparatively shallow channel; but a mathematical analysis of the distortion with frictionless flow does not serve much useful purpose.

SEICHES

350. An accidental tilting of the surface of a deep lake or enclosed sea, such as may be produced by wind, or a variation in the barometric pressure, or by any other cause, often is followed by periodic oscilla-

tions of the surface before it returns to normal level. These oscillations are called *seiches.* Since the currents set up by seiches are never strong, the characteristics of a seiche in a long canal of uniform dimensions, closed at both ends, may be derived from the equations for frictionless flow in such a canal. The frictionless tides and currents in a canal open at the initial end and closed at the other have been derived in equations (251) and (252). If the canal is closed at both ends, the currents at the initial end are zero. Placing $x=0$ in equation (252) the equation of the current at the initial end becomes:

$$v=-(g/c)A_0 \sin (at+\alpha_0) \tan \gamma$$

This current is zero if the speed of the oscillations, a, is such that $\tan \gamma=0$, or if:

$$\gamma=aL/c=\pi$$

whence:

$$a=\pi c/L$$

The period, T, of these oscillations is therefore, from equation (28), paragraph 49:

$$T=2\pi/a=2L/c=2L/\sqrt{gD} \tag{266}$$

Thus the period of free oscillation in a canal 5,000 feet in length, and 30 feet in mean depth is $10{,}000/\sqrt{30g}=322$ seconds$=5\frac{1}{3}$ minutes. If then the entrance were closed, an oscillation of this period, once started, would continue, like the oscillations of a pendulum, until damped out by friction.

At a point distant x from the initial end of the canal, and at the time t, the height of the water surface above its normal level, is found by placing $\gamma=\pi$ in equation (251), and is:

$$y=A_0 \cos (at+\alpha_0) \cos (\pi x/L) \tag{267}$$

and the current is, from equation (252):

$$v=-(g/c)A_0 \sin (at+\alpha_0) \sin (\pi x/L) \tag{268}$$

At the middle of the canal, $x=\frac{1}{2}L$, so that $\pi x/L=\pi/2$; and $y=0$ The middle of the canal is then the node of the oscillation. The currents there reach the maximum amplitude of gA_0/c.

Since the current at the initial entrance also is zero when the oscillations have such a period that $\gamma=2\pi$, 3π, 4π, etc., similar oscillations with periods of one-half, one-third, one-fourth, etc., of the first may be set up in a canal closed at both ends. These have two, three, four, etc., nodal points respectively.

The corresponding oscillations of a deep lake or arm of the sea are more complicated, but those observed at any locality often have one or more fairly well defined periods. In a deep narrow lake with a regular shore line, the observed period of the oscillations usually agrees with the computed period for a canal of equivalent dimensions, closed at both ends. Even when the conformation of the bed and shores of the lake is not so regular, the observed periods may agree with those computed for an equivalent canal extending to a selected point on an opposite shore; but as the selection of the point must be based on the known period, the analogy does not serve much useful purpose.

Seiches are damped out by the frictional resistance of the currents which they produce. For an oscillation of a given amplitude these currents decrease as the depth of the lake increases. Seiches therefore are most marked in deep bodies of water. At many localities on the Great Lakes, seiches of a foot or more occur with such frequency that an allowance is made for them in the design of navigation channels.

Seiches in tidal waters are superimposed upon, and more or less obscured by, the fluctuations of the tide; but the tide gage records at some stations occasionally are marked by saw-toothed irregularities produced by seiches. Their characteristics may be ascertained by taking off their departures from a smooth tide curve. In San Francisco Bay, seiches produced by variations in the barometric pressure and winds have a period of about 45 minutes, and may have a range of as much as a third of a foot. It has been observed that earthquake waves reaching the bay from distant points set up oscillations of the same period.

At some of the ports on the Pacific coast in California, the currents set up by seiches often cause a surging of large vessels at wharves, sometimes with sufficient violence to break the mooring lines. The *surge* is experienced chiefly at the wharves in the less enclosed parts of the harbors. The reason for the prevalence of a troublesome surge in this region is not clear. It is possible that the characteristic periods of the seiches may agree with the period in which a vessel, as customarily moored to a wharf, comes and goes with the stretching and slackening of its lines. The usual remedy is to make fast to the wharf with short, taut lines.

CHAPTER VIII

COMPUTATION OF TIDES AND CURRENTS IN LONG CANALS WHEN THE FLOW IS NOT FRICTIONLESS

CONNECTING CANALS

351. *General considerations.*—Perhaps the most important application of the principles of tidal hydraulics is in the computation of the currents which may be expected in a projected long open connecting canal because of the tides at the entrances; and in the concurrent determination of the elevation of low and high water through the canal. As has been pointed out, no artificial canal can be so deep that the tidal flow approaches a frictionless condition, if the currents are of any consequence whatever. Frictional resistance to flow has therefore an important effect upon the tides and currents. The variation in the depth and width of the water prism as the tide rises and falls also may have a sufficient effect to warrant consideration, and the entrance, recovery and velocity heads may be more than negligible.

352. *Accuracy required.*—The usual purpose of such computations is to ascertain whether the currents will be strong enough to affect adversely the use of the canal for navigation, or to cause serious erosion of the bed and banks. The computations also show the depths to which the successive sections of the channel must be excavated to afford the designed depth for navigation at the selected low-water datum. These purposes are fulfilled if the maximum current ordinarily to be expected in any part of the canal is reliably determined to say the nearest half of a foot per second, and the elevation of low water in the successive sections to the nearest half foot; but good workmanship in the calculations usually requires that they check to the nearest tenth of a foot per second, and tenth of a foot of elevation respectively.

It is well to recollect that the results rest on the selection of the coefficient of friction, and that this depends upon the undeterminable irregularities of the channel as actually excavated. Furthermore, the actual currents will vary from day to day with the varying range and even the varying form of the tides at the entrances. These variations may be intensified by winds and storms, which may pile up the water at one entrance and draw it away from the other. A precise determination of the currents and tides produced by a particular fluctuation of the entrance tides is of academic interest only. The computations call therefore for reliability rather than precision.

353. *Selection of representative entrance tides.*—The computation of the tides and currents in a canal is far too laborious to warrant repetition for each successive tidal fluctuation at the entrances, even if any useful purpose would be served thereby. Representative tidal fluctuations at each entrance should therefore be selected from a study of the actual tidal fluctuations during a month or more.

If the tides at both entrances are of the semidiurnal type, with no large variations between springs and neaps, and are not much deformed by overtides, the representative tide at each entrance may be taken as a simple harmonic fluctuation with the speed of the M_2 component and the amplitude of the mean semirange of the tide. The most convenient origin of time in this case is at a high water at the initial entrance. The initial phase of the tide at this entrance is then zero. The initial phase at the other entrance may be obtained from the difference between the average lunitidal intervals at the two entrances, corrected, if necessary for the differences in longitude (par. 10). The corrected difference, in solar hours, multiplied by the speed of the M_2 component, 28°98 per hour, gives the initial phase of the tide at the far entrance. This phase is positive if the tide at the far entrance is the earlier, and negative if it is later than at the initial entrance. Published data on the lunitidal intervals at stations near the ends of the canal may be based on such a limited number of observations as to have no great weight. If a reliable determination of the lunitidal intervals is not available, the recorded differences in the times at high water at the two entrances, and in the times of low water, over a period of 29 days, or a multiple thereof, should be averaged. A material discrepancy between the average difference in the times of high water at the two entrances to the canal and the average difference in the times of low water, indicates that overtides are of sufficient importance to warrant consideration.

354. If the tides at the entrances are of the same general type as those considered in the preceding paragraph, but the daily tide curves at one or both entrances are so distorted by overtides that they cannot be represented satisfactorily by a simple harmonic fluctuation, either average or composite tide curves may be prepared by the methods

outlined in paragraphs 304 and 305. The tidal heights on these curves should, however, be taken off at *lunar* hourly intervals, beginning generally either at a lunar transit or at the time of high water at the initial entrance to the canal.

355. If the tides at an entrance have a marked variation between springs and neaps, it may be advisable to determine the tides and currents in the canal at mean spring tides, or at ordinary spring tides (par. 181); and perhaps at the corresponding neap tides as well. If the daily tide curves show no marked distortions, the representative spring and neap tides may be taken, without introducing errors which will affect the results materially, as simple harmonic fluctuations with the speed of the M_2 component, and amplitudes of one-half of the spring and neap ranges respectively; otherwise average or composite curves of spring and neap tides may be prepared.

356. Probably the most satisfactory method for dealing with an entrance tide of the mixed type is the preparation of a composite curve whose timing and elevations conform to the times and elevations of mean lower low, higher low, lower high and higher high waters, as outlined in paragraph 306; or a representative tropic tide could be selected from the records.

357. If the entrance tides are of the semidiurnal type, the currents produced by the repetition of a single properly selected average, or spring, or neap semidiurnal fluctuation will afford an adequate indication of the average, spring or neap currents and tides to be expected in the canal. To facilitate the computations, the representative entrance tides of this type should be adjusted as necessary to afford a smooth curve when identically repeated in successive periods of 12 lunar hours. When either or both of the entrance tides are of the mixed or diurnal types, the representative tides similarly should follow curves which are identically repeated every 24 mean lunar hours.

358. The representative tides at the two entrances to a canal should be referred to the same horizontal datum. If either end of the canal takes off from the upper part of a generally shallow bay or river estuary, the mean level of the tides at the two entrances may not be at the same elevation. The same situation may arise if the canal connects two oceans or seas in which the water density and mean meteorological conditions are not the same. Any uncertainty may be removed by connecting the tide stations at the proposed entrances by a line of precise levels. Daily variations due to winds, freshets, and other meteorological causes may produce a constant component of the head between the entrances, of sufficient magnitude to have a marked effect upon the currents. It may, therefore, be advisable to select one or more typical concurrent storm tides at the two entrances, which would produce large differences in the head through the canal,

and determine the currents due to these, as well as those produced by the representative normal tidal fluctuations.

359. *Primary currents and tides in the canal.*—If the selected representative entrance tides are not simple harmonic fluctuations of the same speed, they may be approximated more or less closely by such fluctuations. These may be termed the *primary entrance tides.* The *primary currents and tides* in the canal which would be produced by the primary entrance tides are first computed. These computations are based on the depth and width of the canal at mean tide, and omit the effect of the minor components of the friction term (par. 226). The primary currents and tides may then be adjusted to develop the deformations produced by the minor components of the friction term, by the variation in the width, depth, and area of the cross section of the water prism as the ride rises and falls, and by the entrance, recovery, and velocity heads; and to develop also the variations because of a departure of the representative entrance tides from the simple harmonic fluctuations from which the primary currents and tides were derived.

The primary currents and tides usually afford a fair representation of the currents and tides to be expected in the canal because of the ordinary tidal fluctuations; but their adjustment, although a laborious procedure, may be warranted to give a more complete and assured picture of the anticipated tidal flow. The effect of storm tides can be ascertained only by going through the latter process.

360. *Determination of primary entrance tides.*—The primary tides most nearly conforming to the selected representative tides ordinarily will have the speed of the M_2 component, whose component hour is the mean lunar hour of 1.035 mean solar hours. By taking off from the representative tide curves the heights on 24 successive lunar hours after any assumed origin of time, the amplitude A, and the initial phase, α, of the primary tide at this origin of time, may be computed from equations (56), (57), (47), and (48) developed in Chapter II, viz:

$$12c_2=(h_0 \cos 0+h_1 \cos 30°+h_2 \cos 60°+. \ . \ .+h_{23} \cos 330°) \quad (56)$$

$$12s_2=(h_0 \sin 0+h_1 \sin 30°+h_2 \sin 60°+. \ . \ .+h_{23} \sin 330°) \quad (57)$$

$$\tan \zeta=s_2/c_2 \qquad A=s_2/\sin \zeta=c_2/\cos \zeta \qquad (47)(48)$$

In which h_0, h_1, etc., are the tidal heights at the successive lunar hours, and $\zeta=-\alpha$.

The abbreviation of the computations is explained in paragraph 94.

A consideration of the derivation of these equations shows that if the representative entrance tides are taken as a fluctuation which is

identically repeated every twelve lunar (or other component) hours, the values of c_2 and s_2 may be derived from the equations

$$6c_2 = (h_0 - h_6) \cos 0 + (h_1 - h_7) \cos 30° + \ldots + (h_5 - h_{11}) \cos 150° \quad (56A)$$

$$6s_2 = (h_0 - h_6) \sin 0 + (h_1 - h_7) \sin 30° + \ldots + (h_5 - h_{11}) \sin 150° \quad (57A)$$

The mean tide elevation of the primary entrance tides should be the mean of the elevations computed from the representative tides at the two entrances.

COMPUTATION OF PRIMARY TIDES AND CURRENTS IN A CONNECTING CANAL

361. A general mathematical analysis of the tides and currents in a long canal appears impossible if the friction term in the general equation of motion is taken as a reversing function which varies with the square of the current velocity. A general solution when the frictional resistance is assumed to vary with the first power of the velocity is given by Maurice Lévy in "Leçons sur la Théorie des Marées" (Gauthier-Villars, 1898); and its application to the Cape Cod Canal is presented in a paper by William Barclay Parsons contained in the Transactions of the American Society of Civil Engineers, volume LXXXII (1918), pages 1–157. The equations developed by this analysis are lengthy and unwieldy, and the established coefficients of frictional flow are not applicable thereto. Another method for determining the tides and currents is presented by Col. Earl I. Brown, Corps of Engineers, United States Army, in a paper on the Transactions of the American Society of Civil Engineers, volume 96 (1932), pages 753 et seq. The solution therein presented proposes that the currents at high water be determined from the mean depth of the canal at high water, and the currents at low water from the mean depth at low water.

362. A better method is to compute the primary currents and tides from established frictional coefficients, by a process of successive approximations, on a line of procedure somewhat similar to that applied in computations of steady flow. The canal is divided into subsections so short that the variation in the velocity of the current because of channel storage is not material in any subsection. The primary currents in the subsections, and the primary tides at the ends of the subsections, must satisfy the two conditions:

(*a*) The fluctuations of the current in each subsection must conform to the fluctuations of the surface head set up by the tides at the ends of the subsection.

(*b*) The currents in the subsections must also conform to the storage and release of water from subsection to subsection because of the rise and fall of the tides.

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The computations are started by determining the tides which would be produced if the instantaneous profiles were straight lines. The primary current in the subsection at the middle of the canal is then computed from the surface head established by these tides, and the currents in the other subsections determined therefrom by computing the increments in the velocity due to tidal storage between the successive subsections. The surface heads corresponding to these currents are next computed and adjusted to the total head through the canal. These subsection heads establish corrected primary tides, and corrected instantaneous profiles, more nearly conforming to the true profiles than the straight lines initially assumed. The primary current in the middle subsection is then recomputed from the adjusted surface head in this subsection, and corresponding currents in the other sections from the storage and release of water with the corrected tidal fluctuations. The process is repeated until the further corrections become negligible.

363. It may be noted that the computations are started by taking the instantaneous profiles as straight lines, and the current at the middle of the canal as unaffected by channel storage. As shown in paragraph 331 both of these conditions would be approximately realized in as long a canal as is likely to be undertaken, if frictional resistance were neglected. An examination of the recorded instantaneous profiles in actual canals shows that they do not, in fact, depart widely from straight lines. The procedure will be found applicable to any case likely to be encountered.

364. *Coordinate components of the primary tides and currents.*—The equation of the primary tide at any station on the canal is in the form:

$$y=A \cos (at+\alpha)$$

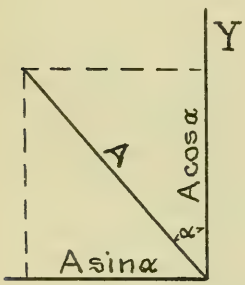

FIGURE 59.—Coordinate components of the tide.

The speed, a, at all stations is that selected for the primary entrance tides, but the initial phase, α, differs from station to station. To carry out the computations outlined in the preceding paragraphs, these tides are resolved into components with common initial phases. As is apparent from figure 59, the primary tide at a given point may be resolved into two components; the Y component with an amplitude of $A \cos \alpha$ and an initial phase of 0°, and the X component with an amplitude of $A \sin \alpha$, and an initial phase of 90°.

The equation of the primary surface head in a subsection of the canal is similarly

$$h_s=H \cos (at+H°)$$

and the surface heads may likewise be resolved into Y components whose amplitudes are $H \cos H°$ and whose initial phases are 0°; and X components whose amplitudes are $H \sin H°$, and whose initial phases are 90°.

Designating the amplitudes of the Y and X components of the tide at the initial entrance as $A_0 \cos \alpha_0$ and $A_0 \sin \alpha_0$ respectively, the amplitudes of the Y and X components of the tide at any point in the canal are:

$$A \cos \alpha = A_0 \cos \alpha_0 + \Sigma H \cos H° \tag{269}$$

$$A \sin \alpha = A_0 \sin \alpha_0 + \Sigma H \sin H° \tag{270}$$

In which $\Sigma H \cos H°$ and $\Sigma H \sin H°$ are respectively the sums of the amplitudes of the Y and X components of the heads in the successive subsections of the canal between the initial entrance and the given point.

365. The primary tides and heads may then be computed in terms of the amplitudes of their coordinate components. After the values of the components have been satisfactorily established, the amplitude and phase of the resultant tide is readily determined from the equations:

$$\tan \alpha = A \sin \alpha / A \cos \alpha, \; A = A \sin \alpha / \sin \alpha = A \cos \alpha / \cos \alpha \tag{271}$$

The quadrant in which α lies is fixed by the algebraic signs of $A \sin \alpha$ and $A \cos \alpha$. A schedule of the values of α corresponding to the value, (α), taken from a table of tangents, is set down for convenient reference.

$A \sin \alpha$	$A \cos \alpha$	α
+	+	(α)
+	−	$180° - (\alpha)$
−	−	$180° + (\alpha)$
−	+	$-(\alpha)$

The primary current at a given point in the canal:

$$v = B \sin (at + \beta)$$

similarly may be resolved into two components, one with the amplitude of $B \sin \beta$ and the initial phase of 0°, and the other with the amplitude of $B \cos \beta$, whose amplitude differs by 90° from the first.

366. *Coordinate amplitudes of the tides for first computation.*—The computations are started with the tides which would be produced if the instantaneous profiles were straight lines.

The equations of the entrance tides are:

$$y_0 = A_0 \cos (at + \alpha_0) \qquad y_1 = A_1 \cos (at + \alpha_1).$$

If the instantaneous profiles were straight lines the equation of the tide at a point in the canal distant x from the initial end would be:

$$A \cos (at+\alpha) = y_0 + (x/L)\ (y_1 - y_0) = A_0 \cos (at + \alpha_0) + (x/L)[A_1 \cos (at+\alpha_0) - A_0 \cos (at+\alpha_0)]. \tag{272}$$

The coordinate amplitudes of the tide at the point x are then:

$$A \sin \alpha = A_0 \sin \alpha_0 + (x/L)\ (A_1 \sin \alpha_1 - A_0 \sin \alpha_0) \tag{273}$$

$$A \cos \alpha = A_0 \cos \alpha_0 + (x/L)\ (A_1 \cos \alpha_1 - A_0 \cos \alpha_0). \tag{274}$$

367. *First computation of primary currents in subsections.*—The primary current in each subsection may be taken as that at the middle of the subsection, and the midpoints of the successive subsections will be designated the *velocity stations.* The amplitude, H, and the initial phase, $H°$, of the head, and the amplitude, S, of the slope in the middle subsection of the canal are computed, as described in paragraph 239, from the components of the tide at the ends of the subsection, derived from equations (273) and (274). The amplitude, B_0, the initial phase, β_0, and the resulting coordinate amplitudes, $B_0 \sin \beta_0$ and $B_0 \cos \beta_0$, of the primary current at the velocity station at the middle of the subsection are then computed by the process set forth in paragraphs 246 and 248.

368. The corresponding primary currents at the other velocity stations are determined by the general equation of continuity (equation 182):

$$\partial Q/\partial x + z\partial y/\partial t = 0.$$

Since differential equations remain approximately true when small finite increments are substituted for the differentials, equation (182) may be written

$$\Delta Q/\Delta x + z\partial y/\partial t = 0$$

or

$$\Delta Q = -z\Delta x \partial y/\partial t. \tag{275}$$

In this equation ΔQ is the algebraic increase, at any instant, in the discharge from one velocity station to the next, z the mean width of water surface between the stations at mean tide and Δx the distance between the sections. It should be noted that Δx may be large when expressed in feet while being small in relation to the change which it produces in the discharge.

Designating the area, $z\Delta x$, of the water surface, at mean tide, between successive velocity stations as U, equation (275) becomes:

$$\Delta Q = -U\partial y/\partial t. \qquad (276)$$

369. Strictly speaking, the rate of rise of water surface, $\partial y/\partial t$, should be computed at the center of gravity of the water surface between the two velocity stations, but its location need not be determined with mathematical precision. If the canal has a constant width at mean tide, the *storage stations*, at which $\partial y/\partial t$ is to be computed, are midway between the velocity stations; and if the subsections are also all of the same length, these storage stations are at the ends of the subsections. If the width of the canal is tolerably constant, the storage stations also may be taken at points half way between the velocity stations; otherwise the location of the center of gravity of the water surface may be roughly estimated and the storage stations selected accordingly.

The equation of the primary tide at a storage station is in the form:

$$y = A \cos (at + \alpha).$$

Differentiating:

$$\partial y/\partial t = -aA \sin (at + \alpha).$$

Substituting this value in equation (276):

$$\Delta Q = aUA \sin (at + \alpha). \qquad (277)$$

370. Designating respectively the area of the cross section of the velocity station at the middle of the canal, at mean tide, as M_0; the area at any other velocity station as M; the discharges at these stations at a given instant as Q_0 and Q; the amplitudes of the currents as B_0 and B; and the initial phases of the currents as β_0 and β; then;

$$Q_0 = M_0 v = B_0 M_0 \sin (at + \beta_0)$$

and

$$Q = Q_0 + \Sigma\Delta Q = B_0 M_0 \sin (at + \beta_0) + \Sigma aUA \sin (at + \alpha). \qquad (278)$$

Since $Q = MB \sin (at + \beta)$, equation (278) becomes, after dividing both members by M,

$$B \sin (at + \beta) = (M_0/M)B_0 \sin (at + \beta_0) + (M_0/M)\Sigma(aU/M_0)A \sin (at + \alpha).$$

The coordinate amplitudes of the current at the velocity station are then:

$$B \sin \beta = (M_0/M)B_0 \sin \beta_0 + (M_0/M)\Sigma(aU/M_0)A \sin \alpha \qquad (279)$$

$$B \cos \beta = (M_0/M)B_0 \cos \beta_0 + (M_0/M)\Sigma(aU/M_0)A \cos \alpha. \qquad (280)$$

371. Placing for brevity:

$$M_0/M=m \tag{281}$$

$$aU/M_0=I \tag{282}$$

equations (279) and (280) may be written:

$$(B/m)\sin\beta=B_0\sin\beta_0+\Sigma IA\sin\alpha \tag{283}$$

$$(B/m)\cos\beta=B_0\cos\beta_0+\Sigma IA\cos\alpha. \tag{284}$$

In equations (283) and (284), m is the ratio of the areas, at mean tide, of the cross section at the middle velocity station to that at the given station, $A\sin\alpha$ and $A\cos\alpha$ are the coordinate amplitudes of the tides at the intervening storage stations, and the values of I are determined from the surface areas, U, at mean tide, between the successive intervening velocity stations and the speed, a, of the primary tidal fluctuations, whose usual value is 0.0001405 radians per second.

372. If the canal has a uniform width and cross section, and consequently a uniform mean depth, D, at mean tide; $m=1$, and $I=az\triangle x/M_0=a\triangle x/D$. If, further, all of the sections are of the same length, the value of I is the same for all.

The computations indicated in equations (283) and (284) establish the first values of the amplitudes B, and the initial phases, β, of the currents in the subsections of the canal.

373. *Heads corresponding to computed velocities in subsections.*—The relations between the amplitude, B, of the primary current in a short section of channel of length, l; its angular lag, ϕ, and the amplitude, S, of the slope in the section, have been developed in chapter V. From equation (153) in that chapter:

$$\tan\phi=(3\pi/8)(a/g)C^2r/B. \tag{285}$$

It is convenient to place:

$$p=(3\pi/8)(a/g)C^2r \tag{286}$$

in which a is the speed of the fluctuation, in radians per second, $g=32.16$, and C and r the Chezy coefficient and hydraulic radius in the section at mean tide.

If the speed of the primary tidal and current fluctuations has its usual value of 0.0001405 radians per second, equation (286) becomes:

$$p=0.000{,}005{,}148\ C^2r. \tag{287}$$

The logarithm of the coefficient is 4.71160−10.

From equations (285) and (286):

$$\tan\phi=p/B. \tag{288}$$

Again, from equation (151):

$$S \sin \phi = aB/g$$

whence:

$$H = lS = l(a/g)B/\sin \phi. \tag{289}$$

If $a=0.0001405$ and $g=32.162$, the value of a/g is 0.000,00437 and its logarithm is $4.64042-10$.

The relation between the initial phase, $H°$, of the head and the initial phase, β, of the primary current is, from equation (150):

$$H° = \beta + \phi + 90°. \tag{290}$$

The values of H and $H°$, derived from equations (288), (289), and (290) give the values of the coordinate amplitudes, $H \cos H°$ and $H \sin H°$, of the heads in the subsections corresponding to the first computation of the currents.

374. *Corrected tides and velocities.*—The computed coordinate amplitudes of the heads in the subsections are so adjusted that their sums are equal to the differences between the coordinate amplitudes of the tides at the entrance. The corrected coordinate amplitudes of the tides at the ends of the subsections produced by the adjusted coordinate amplitudes of the heads are computed from equations (269) and (270). The coordinate amplitudes, $B_0 \sin \beta_0$ and $B_0 \cos \beta_0$, of current in the middle subsection are next recomputed from the adjusted head in the subsection, but the recomputation may be somewhat abbreviated. Let H and H' be the initial and adjusted values of the amplitude of the head, and $S=H/l$ and $S'=H'/l$ the corresponding values of the amplitude of slope in the subsection. The ratio of the corrected value of $P'=1.0854\ C\sqrt{rS'}$ (par 245) to the value, $P=1.0854\ C\sqrt{rS}$, initially computed, is:

$$P'/P = \sqrt{S'}/\sqrt{S} = \sqrt{H'}/\sqrt{H}$$

So that:

$$\log P' = \log P + \tfrac{1}{2}(\log H' - \log H) \tag{291}$$

and similarly:

$$\log (P'/S') = \log (P/S) - \tfrac{1}{2}(\log H' - \log H). \tag{292}$$

The corrected values of ϕ, B_0 and β_0 are determined from the corrected values of P'/S' and P' as explained in paragraph 246.

The currents in the other subsections are then recomputed from equations (283) and (284). In applying these equations, the coordinate amplitudes of the tide at any storage station which does not coincide with the end of a subsection are interpolated between the corrected values at the ends of the subsection in which it lies, on the assumption that the instantaneous profiles in each subsection are sub-

stantially straight lines. The corresponding heads are then recomputed and the process repeated until a satisfactory concordance of the currents and heads is attained.

375. *First example.*—The computations of the primary tides and currents in a canal of uniform dimensions will be illustrated by applying the procedure outlined in the preceding paragraphs to a canal 200,000 feet (37.8 miles) in length, and of uniform cross section, with a bottom depth of 40 feet at mean tide, a bottom width of 250 feet, and side slopes of 1 on 3⅛ (fig. 60).

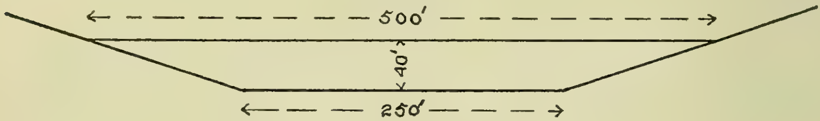

FIGURE 60.—Cross section of assumed canal.

The representative tide at the initial end of the canal has an amplitude of 4 feet and the speed of the M_2 component (28°.98 per mean solar hour, or 30° per mean lunar hour). The representative tide at the other entrance has an amplitude of 2 feet, and the same speed. Its high water occurs 2 lunar hours, or 60°, before that at the initial entrance. Taking the origin of time at high water at the initial entrance, the equation of the tide at this entrance is then $y=4 \cos m_2 t$ and at the other entrance, $y=2 \cos (m_2 t+60°)$.

The area of the cross section of the water prism at mean tide is 15,000 square feet and the surface width is 500 feet, giving a mean depth of 30 feet at mean tide. The hydraulic radius at mean tide is also taken as 30 feet, as the refinement of computing the wetted perimeter is superfluous in view of the uncertainty in the Chezy coefficient. The Chezy coefficient at mean tide is taken as 120.

376. *Division into subsections.*—The canal will be divided into a middle subsection and two subsections on either side, total of 5 subsections, each 40,000 feet in length, as shown in figure 61.

FIGURE 61.—Division of canal into subsections.

A canal of uniform dimensions should always be divided into an odd number of sections each of the same length; but shorter subsections are required in a shallower canal. The ends and midpoints of the subsections are conveniently indicated by station numbering, as shown in the figure, the stations being taken as 1,000 feet in length. In the present example the velocity station at the middle of the canal is at station 100, and the velocity stations of the other subsections at stations 20, 60, 140, and 180. The storage stations

coincide with the ends of the subsections, at stations 40, 80, 120 and 160.

377. *Initial component tides at the storage stations.*—These are computed from equations (273) and (274):

$$A_0=4 \qquad \alpha_0=0 \qquad A_1=2 \qquad \alpha_1=60°$$
$$A_1 \sin \alpha_1=1.732 \qquad A_1 \cos \alpha_1=1.0$$
$$A_0 \sin \alpha_0=0 \qquad A_0 \cos \alpha_0=4.0$$
$$1.732 \qquad —3.0$$

Initial component tides

(1)	(2)	(3)	(4)	(5)	(6)
Station	x/L	$1.732\,x/L$	$-3\,x/L$	$A \sin \alpha = A_0 \sin \alpha_0 + (3)$	$A \cos \alpha = A_0 \cos \alpha_0 + (4)$
0	0	0	0	0	4.0
40	.2	.346	−.6	.346	3.4
80	.4	.693	−1.2	.693	2.8
120	.6	1.039	−1.8	1.039	2.2
160	.8	1.386	−2.4	1.386	1.6
200	1.0	1.732	−3.0	1.732	1.0

378. *Coordinate currents in middle subsection.*—The head in the middle subsection is computed from the component tides at stations 80 and 120.

$$H \sin H°=A \sin \alpha_{120}-A \sin \alpha_{80}=1.039-0.693=0.346$$
$$H \cos H°=A \cos \alpha_{120}-A \sin \alpha_{80}=2.2-2.8=-0.600$$

$$\tan H°=0.346/(-0.600)=-0.577 \qquad H°=180°-30°=150°$$
$$H=0.6/\cos 30°=0.693$$

Then $S=H/l=0.693/40{,}000=0.00001732$. And from the given data:

$$r=30 \qquad C=120 \qquad rS=0.0005196$$

The coordinate components of the current in the middle subsection are computed as described in paragraphs 246 and 248:

$$\log rS=6.71567-10 \qquad \log\sqrt{\cos \phi}=9.93987-10$$
$$\log\sqrt{rS}=8.35783—10 \qquad \log P=\ .47260$$
$$\log 1.0854=\ .03559 \qquad \log B_0 \ =\ .41247$$
$$\log C=2.07918 \qquad B_0 \ =2.580$$

$\log P=\ .47260$ From equation (162):

$\log S=5.23855-10$ $\quad \beta_0=H°-\phi-90°$

$\log P/S=5.23405$ $\quad =150°-40°40'-90°$

From table IX: $\phi=40°.7=40°40'$ $\quad =19°20'$

$$B_0 \cos \beta_0=2.440$$
$$B_0 \sin \beta_0=0.854$$

379. *Primary currents in other subsections.*—The amplitudes and phases of the currents which would be produced at the other velocity stations under the initial assumption are computed from equations (283) and (284). These computations are tabulated in figure 62, facing page 198. The coordinate components of the tide at the entrances and at the storage stations, found in paragraph 377, are entered in columns (2) and (3). Since the canal has a constant cross section.

$$I = a\Delta x/D = 0.000{,}140{,}5 \times 40{,}000/30 = 0.187.$$

This value is entered in column (4). A much larger value of I would indicate that the canal should be divided into shorter subsections. The resulting coordinate velocity increments, $IA \sin \alpha$ and $IA \cos \alpha$, are entered in columns (5) and (6). For these and for the subsequent computations, the slide rule affords satisfactory accuracy.

The coordinate amplitudes, $B_0 \sin \beta_0$ and $B_0 \cos \beta_0$, of the current at station 100, determined in paragraph 378, are entered opposite this station in columns (7) and (8). The values of $(B/m) \sin \beta$ and $(B/m) \cos \beta$ at station 60 are found by subtracting, algebraically, the coordinate velocity increments $IA \sin \alpha$ and $IA \cos \alpha$ at station 80 from the coordinate amplitudes of the currents at station 100 (since the summation is in the negative direction); and the values at station 20 by subtracting the increments at station 40 from the values found at station 60. The values of $(B/m) \sin \beta$ and $(B/m) \cos \beta$ at station 140 are similarly found by adding, algebraically, the coordinate increments at station 120 to the coordinate amplitudes at station 100; and at station 180 by again adding the increments at station 140. The totals of columns (5) and (6) may be checked against the differences between the last and first lines of columns (7) and (8) respectively.

The values of $\tan \beta$, from columns (7) and (8) are entered in column (9), and the corresponding values of β in column (10). Since, in the present example, the area of the cross section of the canal is the same at all stations, $m=1$ and $B/m=B$. The entries in columns (11) and (12) therefore are omitted, and the amplitudes, B, of the current at the velocity stations entered in column (13) from the relation:

$$B = (B/m) \sin \beta/\sin \beta = (B/m) \cos \beta/\cos \beta.$$

The values of β and B at station 100 derived in columns (10) and (13) should check with those found in paragraph 378.

380. *Surface heads corresponding to computed currents in subsections.*—The computation is continued in columns (14) to (24) of figure 62. The value of p, column (14) for all subsections is, from equation (287):

$$p = 0.000{,}005{,}148 \times \overline{120}^2 \times 30 = 2.224.$$

Dividing by B, the values of $\tan \phi$ (equation 288) are entered in column (15) and the corresponding values of ϕ in column (16). The computed value of ϕ at station 100 should check with that derived in paragraph 378.

The value of la/g, column (17), for all subsections, is (par. 373):

$$la/g=0.000,004,37\times 40,000=0.175.$$

The values of Bla/g are entered in column (18) and those of $H=(Bla/g)/\sin \phi$, from equation (289), in column (19). The values of $H°$, derived from the computed values of β and ϕ, are entered in column (20), and the corresponding computed coordinate amplitudes $H \sin H°$ and $H \cos H°$ in columns (21) and (22). The computed values at station 100 should check with those found in paragraph 378. The initially computed values at this station should be entered in these columns, even if minor inaccuracies have produced slight differences in this check computation. A material difference would indicate the need for reviewing the entire work.

381. *Adjustment of coordinate heads.*—As is to be expected, the sums of the computed coordinate amplitudes of the surface heads in the subsections, in columns (21) and (22) differ by small residuals from the total coordinate amplitudes of the heads between the entrances shown in columns (2) and (3). The computed values are adjusted in columns (23) and (24), by dividing the residuals as equally as may be between them, so that the sums agree with the actual coordinate amplitudes of the head between the entrances.

382. *Recomputation of tides, currents, and heads.*—The corrected coordinate amplitudes of the tides at the storage stations, which in this case are at the ends of the subsections, are next recomputed in columns (2) and (3) of figure 62, by successively adding the adjusted values of $H \sin H°$ and $H \cos H°$, found from the first computation, to the coordinate amplitudes of the tide at the initial entrance. The coordinate amplitudes of the tide at the further entrance, station 200, afford a check on the results. It will be seen that the corrected coordinate amplitudes of the tides at the storage stations differ materially from those used in the first computation. The currents and the resultant heads are therefore recomputed as shown in figure 62 from the corrected data. In the present example, the adjusted coordinate amplitudes of the head in the middle subsection, station 100, in columns (23) and (24) of the initial computation, differ so little from their original values, in columns (21) and (22) that a recomputation of the coordinate amplitudes of the currents in the middle subsection is unnecessary.

383. *Final computation.*—The values of $H \cos H°$ derived from the second computation are practically the same as those determined from the first. While the recomputed values of $H \sin H°$ also are in sufficient concordance to be reasonably acceptable, a third computation, as shown in figure 62, affords a desirable check. In this final computation the primary currents at the entrances to the canal are derived by correcting those at the adjacent velocity stations for the intervening storage. The primary current at the initial entrance, station 0, is derived from that at station 20 by subtracting the increment of the velocity due to the fluctuation of current at station 10. The values of $A \sin \alpha$ and $A \cos \alpha$ at station 10 are obtained by interpolation between their values at station 0 and station 40. For the half subsection, 0 to 20, $\Delta x = 20{,}000$ and $I = 0.0935$. The primary current at the other entrance, station 200, is similarly derived from that at station 180, by correcting for the storage due to the tidal fluctuations at station 190.

384. *Summary of computations.*—The results of the computations are tabulated below. The values of $A \sin \alpha$ and $A \cos \alpha$, derived from the final computations of the subsection heads, are shown in columns (2) and (3), and the resulting amplitudes A, and initial phases α of the tides at the ends of the subsections in columns (6) and (5). Those at the middle of the canal are inserted by interpolation. The amplitudes, B, and initial phases, β, of the primary currents are the final values found in figure 62.

Summary

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Station	$A \sin \alpha$	$A \cos \alpha$	$\tan \alpha$	α	A	B	β
0	0	4.000	0	0	4.00	1.14	47°20′
20	------	-----	------	-------	----	1.41	36°10′
40	−.014	3.703	−.003	−0°10′	3.70	----	------
60	------	-----	------	-------	----	2.01	24°20′
80	+.134	3.246	+.043	+2°30′	3.24	----	------
100	.306	2.944	.104	6° 0′	2.93	2.58	19°20′
120	.478	2.642	.181	10°20′	2.68	----	------
140	------	-----	------	-------	----	3.08	17°50′
160	1.022	1.899	.540	28°20′	2.17	----	------
180	------	-----	------	-------	----	3.48	19° 0′
200	1.732	1.000	.576	30° 0′	2.00	3.65	20°30′

This summary affords the data for writing the equations of the primary currents and tides at the stations. Thus the equation of the primary current at station 0 is:

$$v = 1.14 \sin (m_2 t + 47°20'),$$

and of the primary tide at station 40:

$$y = 3.70 \cos (m_2 t - 0°10').$$

Primary Tides and Currents.

1st Example – Canal 200,000 ft long, 30 ft mean depth

	1	2	3	4	5	6	7	8	9	10		11	12	13	14	15	16		17	18	19	20		21	22	23	24
	Sta.	Asinα	Acosα	I	IAsinα	IAcosα	$\frac{B}{m}\sin\beta$	$\frac{B}{m}\cos\beta$	tan β (7)/(8)	β		$\frac{B}{m}$	m	B	p	tan φ p/B	φ		la/g	Bla/g	H (18)/sin φ	H° β+φ+90		Computed. H sin H°	H cos H°	Adjusted. H sin H°	H cos H°
1										°	'	Initial Computation					°	'				°	'				
2	0	0	4.00																								
3	20						0.659	1.280	0.514	27	10			1.44	2.224	1.544	57	10	0.175	0.252	0.300	174	20	0.03	-0.30	0.04	-0.30
4	40	0.35	3.40	0.187	0.065	0.636																					
5	60						0.724	1.916	0.378	20	40			2.05		1.085	47	20		0.359	0.487	158	0	0.18	-0.45	0.19	-0.45
6	80	0.69	2.80		0.130	0.524																					
7	100						0.854	2.440	0.350	19	20			2.58		0.862	40	40		0.451	0.693	150	0	0.34	-0.60	0.35	-0.60
8	120	1.04	2.20		0.194	0.411																					
9	140						1.048	2.851	0.367	20	10			3.04		0.731	36	10		0.531	0.900	146	20	0.50	-0.75	0.51	-0.75
10	160	1.39	1.60		0.259	0.299																					
11	180						1.307	3.150	0.415	22	30			3.42		0.650	33	0		0.598	1.100	145	30	0.62	-0.91	0.64	-0.90
12	200	1.73	1.00																			Sums		1.67	-3.01	1.73	-3.00
13	200-0	1.73	-3.00	Sums	0.648	1.870	(-0.659)	(-1.280)																1.73	-3.00		
14							648	1.870													Residuals			0.06	0.01		
15												Recomputation															
16	0	0	4.00																								
17	20						0.804	1.141	0.703	35	10			1.40	2.224	1.590	57	50	0.175	0.245	0.290	183	0	-0.02	-0.29	-0.02	-0.29
18	40	0.04	3.70	0.187	0.007	0.692																					
19	60						0.811	1.833	0.442	23	50			2.01		1.110	48	0		0.352	0.475	161	50	+0.15	-0.45	+0.15	-0.45
20	80	0.23	3.25		0.043	0.607																					
21	100						0.854	2.440	0.350	19	20			2.58		0.862	40	40		0.451	0.693	150		0.34	-0.60	0.34	-0.60
22	120	0.58	2.65		0.108	0.496																					
23	140						0.962	2.936	0.328	18	10			3.10		0.720	35	50		0.543	0.930	144	0	0.55	-0.75	0.55	-0.75
24	160	1.09	1.90		0.204	0.356																					
25	180						1.166	3.291	0.355	19	30			3.52		0.634	32	20		0.616	1.159	141	50	0.72	-0.91	0.71	-0.91
26	200	1.73	1.00				(-0.804)	(-1.141)														Sums		1.74	-3.00	1.73	-3.00
27				Sums	0.362	2.150	0.362	2.150																1.73	-3.00		
28																					Residuals			-0.01	0		
29												Final Computation															
30	0	0	4.00				0.833	0.768	1.081	47	20			1.14													
31	10	0	3.93	0.0935	0	0.368																					
32	20						0.833	1.136	0.732	36	10			1.41	2.224	1.580	57	20	0.175	0.247	0.294	183	30	-0.018	-0.294	-0.014	-0.297
33	40	-0.02	3.71	0.187	-0.003	0.694																					
34	60						0.830	1.830	0.452	24	20			2.01		1.110	48	0		0.352	0.475	162	20	+0.144	-0.453	+0.148	-0.457
35	80	+0.13	3.26	0.187	+0.024	0.610																					
36	100						0.854	2.440	0.350	19	20			2.58		0862	40	40		0.451	0.693	150	0	0.340	-0.600	+0.344	-0.604
37	120	0.47	2.66	0.187	0.088	0.497																					
38	140						0.942	2.937	0.321	17	50			3.08		0.724	36	0		0.539	0.915	143	50	0.540	-0.739	+0.544	-0.743
39	160	1.02	1.91	0.187	0.190	0.357																					
40	180						1.132	3.294	0.344	19	0			3.48		0.640	32	40		0.610	1.140	141	40	0.705	-0.895	+0.710	-0.899
41	190	1.55	1.23	0.0935	0.145	0.115																Sums		1.711	-2.981	1.732	-3.000
42	200	1.73	1.00				1.277	3.409	0.375	20	30			3.65										1.732	-3.000		
43				Sums	0.444	2.641	(-0.833) 0.444	(-0.768) 2641													Residuals			0.021	0.019		

FIGURE 62.

192750—40 (Face p. 198)

The primary tides and currents at the entrances and at the middle of the canal are shown in figure 73, page 218.

385. *Shape of instantaneous profiles.*—The characteristic shapes of the instantaneous profiles developed by the successive approximations is readily shown by plotting the values of A sin α and A cos α successively derived in figure 62, since A cos α is the tidal height at a station at 0 hour and $A \sin \alpha = A \cos (-90° + \alpha)$ is that at -3 lunar hours. The instantaneous profiles at 0 hour are shown by the lines marked 0—0 in figure 63. The differences between the first and second

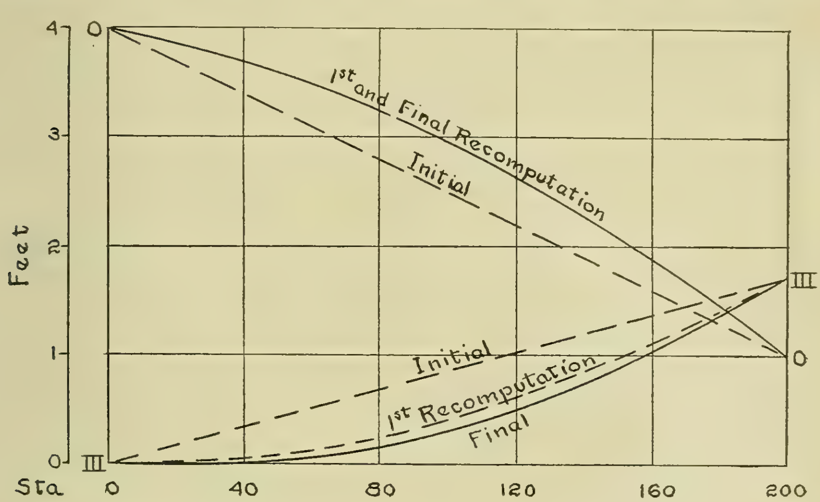

FIGURE 63.—Instantaneous profiles at 0 and III hour.

recomputations are too small to be distinguishable. The successive determinations at -3 lunar hours are shown by the lines marked III—III on the figure.

386. *Sufficiency of subdivision.*—The question may arise whether subsections 40,000 feet in length, in a canal having a mean depth of 30 feet, are sufficiently short to afford a reliable determination of the currents. An independent computation, based on a subdivision into 9 subsections, each 22,222 feet in length, affords the following comparative figures in the equations of the currents at the entrances:

	Computed from 9 subsections	Computed from 5 subsections
At initial entrance	$v=1.12 \sin (at+47°10')$	$v=1.14 \sin (at+47°20')$.
At other entrance	$v=3.65 \sin (at+20°40')$	$v=3.57 \sin (at+20°50')$.

It is apparent, therefore, that the results are but little improved by using the shorter sections.

387. *Effect of frictional resistance.*—The effect of frictional resistance upon the primary tides and currents in the canal selected for the first example is shown by a comparison between the results of the preceding computations and of those derived in Par. 328, Chapter VII, for frictionless flow in the same canal with the same entrance tides. The origins of time and distance are the same in the two cases. Angles are written to the nearest 10 minutes of arc.

	Frictionless flow	Friction considered
VELOCITIES		
At initial entrance	$v=2.99 \sin (m_2t-49°40')$	$v=1.14 \sin (m_2t+47°20')$.
At middle	$v=4.10 \sin (m_2t-30°)$	$v=2.58 \sin (m_2t+19°20')$.
At further entrance	$v=4.67 \sin (m_2t-17°30')$	$v=3.65 \sin (m_2t+20°30')$.
TIDE		
At middle of canal	$y=2.94 \cos (m_2t+19°10')$	$y=2.96 \cos (m_2t+6°)$.

As is to be expected, frictional resistance reduces considerably the amplitude of the currents. Its effect upon the timing of the currents is even more marked. When frictional resistance is neglected, the strength of the positive current at the initial entrance was found to be $(90°+49°40')/28°.98=4.82$ solar hours after high water, while when frictional resistance is considered, the strength of the positive current is $(90°-47°20')/28°.98=1.47$ hours after high water. Frictional resistance advances therefore the strength of the current at the initial entrance by 3½ hours. At the other entrance, it advances the current by 1⅓ hours.

In the present example, the amplitude of the tide at the middle of the channel is but little affected by frictional resistance, but the relative phases of the tide at this station show that the time of high water is altered by nearly half an hour.

388. *Relation between channel storage and the primary currents and discharges at the entrances to a canal.*—The primary discharge at an entrance to the canal is derived immediately by multiplying the current velocity by the area of the cross section at mean tide. Obviously, the difference in the discharges at the two entrances at any moment must be equal to the rate of storage or release of water in the tidal prism of the canal at that moment.

In the present example, the area of the cross section at both entrances is 15,000 square feet. The entrance velocities and discharges and the rates of filling and emptying of the canal during successive tidal cycles are shown diagrammatically in figure 64.

389. The discharge at the initial entrance, station 0, is an inflow into the canal when positive, and an outflow when negative; while at the further entrance, station 200, it is an outflow when positive and an inflow when negative. During the time interval marked *AB*

on the diagram the discharge is in the positive direction at both entrances, and the outflow at station 200 largely exceeds the inflow at station 0. The tidal prism in the canal is therefore emptying through the further entrance. During the interval *BC* it is emptying through both entrances; and during the brief period *CA′* through

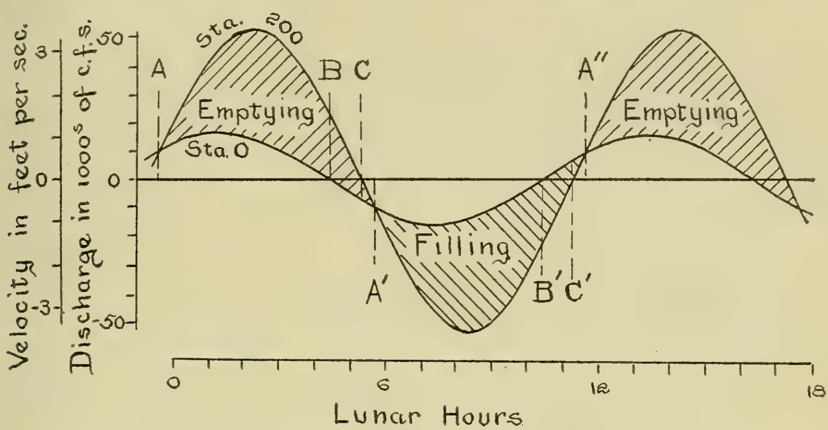

FIGURE 64.—Discharges and storage, canal 200,000 feet long.

the initial entrance. The filling of the tidal prism during the interval *A′A″* follows a similar sequence.

It is apparent from the figure that, in this case, the discharges and velocities at the further end of the canal are caused principally by the filling and emptying of the tidal prism. Because the filling and emptying is largely through this entrance, the currents are there the strongest, although the tidal range is the least.

390. The strength and timing of the currents in a connecting canal depend upon the relative timing of the tides at the entrances, as well as upon the amplitudes of these tides. Thus if in the example developed in the preceding paragraphs, high water at the further entrance is taken 2 lunar hours (60°) *after*, instead of 2 lunar hours before, that at the initial entrance, the equations of the current are found to be:

At the initial entrance: $v=3.04 \sin (at+108°30')$.
At the further entrance: $v=2.30 \sin (at+52°20')$.

The change in the timing of the tide produces therefore the strongest currents at the initial entrance, where the tidal range is the greatest, instead of at the further entrance, where the range is the least. A diagram of these velocities, and consequent discharges and the storage and release of water in the tidal prism of the canal, is shown in figure 65, page 202.

In this case the interval, *AB*, during which the prism is emptying because of the greater outflow through the further entrance, station 200, is shorter than the interval *CA′*, during which the outflow

through the initial entrance is the greater; and the interval *BC*, during which the prism is emptying through both entrances, is relatively long. The tidal prism fills and empties through both entrances, but somewhat more water enters and leaves through the initial entrance than through the other.

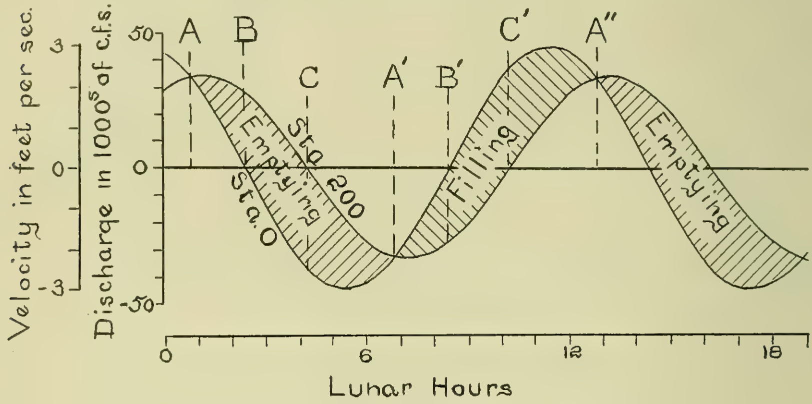

FIGURE 65.—Discharges and storage, with different timing of entrance tides.

391. The longer a canal the less is the net slope set up by given tides at the entrances, and the more are the currents determined by the storage and release of water in the canal prism. It may be expected that in a very long canal the tidal prism will fill and empty from both ends during most of the tidal cycle, and that the currents will be stronger at the two entrances than in the interior of the canal. Thus a computation of the primary currents in a canal 360,000 feet (68.5) miles in length, and 30 feet in mean depth at mean tide, with the same entrance tides as in the first example, i. e.,

$$y_0 = \cos 4\mathrm{m}_2 t \qquad y_1 = 2 \cos (\mathrm{m}_2 t + 60°)$$

and with the same coefficient of roughness, shows the strength of the current decreasing from 2.3 feet per second at the initial entrance to 1.0 foot per second at a point 100,000 feet from that entrance; thence increasing to 3.8 feet per second at the further entrance. The equations of the entrance tides are:

$$\text{Initial entrance: } v = 2.3 \sin (\mathrm{m}_2 t + 149°).$$
$$\text{Further entrance: } v = 3.8 \sin (\mathrm{m}_2 t + 10°50').$$

The diagram of the entrance velocities, discharges, and channel storage, figure 66, shows that in this case the entrance currents are due principally to the filling and emptying of the tidal prism through both ends of the canal.

It further may be expected that in an even longer canal the currents would become so small at some point near the middle of the canal that the tides could be said to meet.

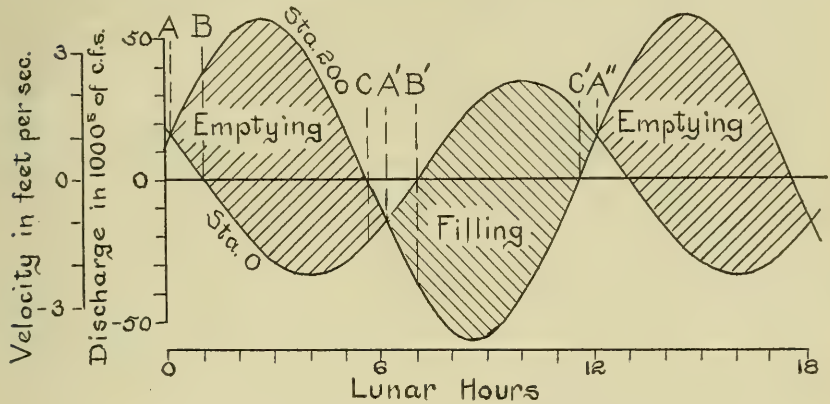

FIGURE 66.—Discharges and storage, canal 360,000 feet long.

392. *Second example.*—The computations of the tides and currents in a connecting canal whose width and cross section is not uniform throughout may be illustrated by applying them to the Chesapeake and Delaware Canal, connecting the estuary of the Delaware River with the head of Chesapeake Bay, after it had been converted into a sea level canal for barge traffic but before it was enlarged to accommodate ocean shipping. The canal then had a horizontal bottom at a project depth of 12 feet below Delaware River low water datum, a designed bottom width of 150 feet from the 12 foot depth contour in the Delaware (station 3+400) to station 10+300, a distance of 9,900 feet; and of 90 feet thence to Back Creek, a tributary of Chesapeake Bay, at station 77+000; a distance of 66,700 feet. Between Reedy Point Bridge, station 9+780, and Biddles Point, station 19+600, the canal was bordered by wide marshes having a large tidal storage. A second outlet to the Delaware, with a depth of 6 feet below datum, and a bottom width of 50 feet, entered the canal near Biddles Point. Some tidal storage outside of the prism proper extended to the deep cut beginning at about station 50, but thence almost to station 77 the only tidal storage was in the prism of the canal. The upper part of Back Creek afforded a tidal basin inside of the bridge at station 77. The canal was designed with side slopes of one on two, but as excavated the cross section generally was somewhat in excess of that projected.

393. *Representative entrance tides selected.*—To afford a comparison between the computed and measured currents, the representative entrance tides will be taken as the recorded tides at stations 5+000

and 77+000 on November 27–28, 1928 when tide and current measurements were made at these stations, at Biddles Point (station 19+600) and at Summit Bridge (station 51+200). The day selected is one on which the tides had little diurnal variation. The origin of time is taken at 7 a. m. on November 27, when the record of the observations begins.

394. *Primary entrance tides.*—The amplitude, initial phase, and mean elevation of the primary tide at station 5+000, are computed in the following tabulation from the recorded tidal heights at this station, by the method explained in paragraph 360.

Primary entrance tide, station 5+000

(1) Lunar hour	0	1	2	3	4	5	6	7	8	9	10	11
(2) Time	7.00	8.04	9.07	10.11	11.14	12.18	13.21	14.25	15.28	16.31	17.35	18.38
(3) Tide	.6	2.4	3.9	5.0	5.5	5.0	3.5	2.3	1.2	.5	−.2	−.7
(4) Lunar hour	12	13	14	15	16	17	18	19	20	21	22	23
(5) Time	19.42	20.45	21.49	22.52	23.56	0.59	1.63	2.66	3.70	4.73	5.77	6.80
(6) Tide	1.2	2.7	4.1	5.1	5.5	4.7	3.2	2.2	1.1	.3	−.2	+.3
(7) (3)+(6)	1.8	5.1	8.0	10.1	11.0	9.7	6.7	4.5	2.3	.8	−.4	−.4
(8) h_6 to h_{11}	6.7	4.5	2.3	.8	−.4	−.4						
(9) (7)−(8)	−4.9	+.6	+5.7	+9.3	+11.4	+10.1						

The mean solar time in lines (2) and (5) corresponding to the lunar hours in lines (1) and (4) is derived by successively adding the length of the lunar hour, 1.035, to the initial time at zero hour. The representative tidal heights, taken in this case from a plot of the recorded tidal heights, on mean solar time, are entered in lines (3) and (6) and summed in line (7). The last six entries are subtracted from the first six in lines (8) and (9). The values of s_2 and c_2 and of A_0 and ζ are then computed from the equations:

$$12s_2=h_0 \sin 0+h_1 \sin 30°+.\ .\ .h_5 \sin 150°.$$
$$12c_2=h_0 \cos 0+h_2 \cos 30°+.\ .\ .h_5 \cos 150°.$$

Angle	Sin	h	Products +	Products −	Cos	Products +	Products −
0	0	−4.9	0		1.00		4.90
30°	.500	+.6	0.30		.866	0.52	
60°	.866	+5.7	4.94		.500	2.85	
90°	1.000	+9.3	9.30		0	0	
120°	.866	+11.4	9.87		−.500		5.70
150°	.500	+10.1	5.05		−.866		8.69

$12s_2=29.46$
$s_2=2.45$

3.37 19.29
3.37

$12c_2=-15.92$
$c_2=-1.33$

$\tan \zeta=12s_2/12c_2=-1.85.$
$\zeta=180°-61°40'=118°20'.$
$A_0=s_2/\sin \zeta=2.79.$
$H_0=\Sigma h/24=2.48.$

The sum of the tidal heights in line (7) divided by 24 gives the mean tide elevation, H_0, above the tidal datum, which in this case is Delaware River low water datum. The equation of the primary entrance tide at station 5+000 is then:

$$y = H_0 + A_0 \cos (at - \zeta).$$
$$= 2.48 + 2.79 \cos (at - 118°20').$$

395. The equation of the primary tide at station 77+000, derived by the same procedure, is:

$$y = 2.79 + 1.45 \cos (at - 73°50').$$

The computed primary tides and the recorded tidal heights at the two stations are plotted in figure 67. They show a satisfactory concordance.

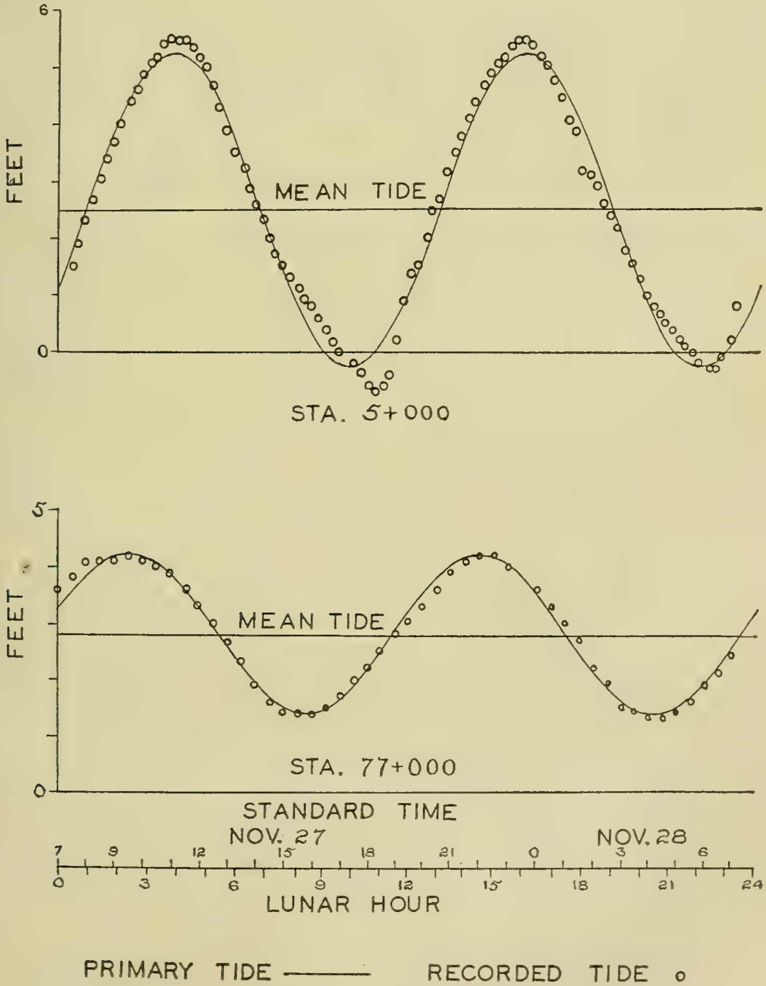

FIGURE 67.—Primary and observed entrance tides, Chesapeake and Delaware Canal.

396. *Mean sea level of primary tides.*—The mean sea level at the two entrances as found in the preceding paragraph differs by 0.31 feet. Long period observations show a similar difference. For the computation of the primary currents and tides the elevation of mean sea level is taken as the mean of the elevations at the entrances, or 2.63 feet above Delaware River datum. The depth of the bottom of the canal is then 14.63 feet throughout.

397. *Division into subsections.*—A subdivision based on the variation in the cross section and in the storage areas is shown in figure 68; station 5+000 being selected as the initial entrance.

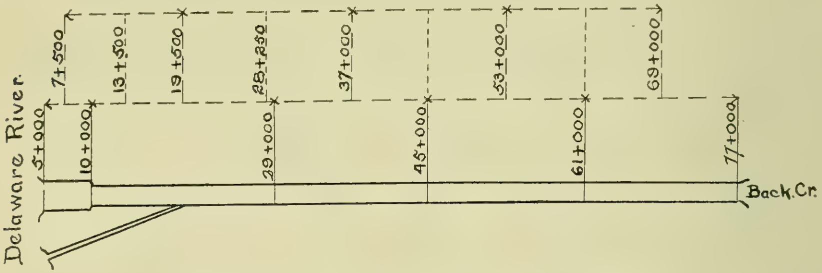

FIGURE 68.—Diagram of Chesapeake and Delaware Canal.

The subsections and their constants are:

1		2	3	4	5	6	7	8	9	10
Subsection (station to station)		Length l	Velocity station	Area M	$m=M_0/M$	Width z	r	C	p	la/g
5+000	10+000	5,000	7+500	3080	0.78	260	11.8	94	0.537	0.022
10+000	29+000	19,000	19+500	2400	1.00	190	12.6	95	.585	.083
29+000	45+000	16,000	37+000	2400		190	12.6	95	.585	.070
45+000	61+000	16,000	53+000	2400	1.00	170	14.1	97	.683	.070
61+000	77+000	16,000	69+000	2400	1.00	170	14.1	97	.683	.070
		72,000								

The velocity stations, column (3), are the midpoints of the subsections. Station 37, nearest the middle of canal, is taken as the base station for velocities. The cross section areas, M, column (4), and the surface widths of the canal, z, column (6), are at mean tide, 2.63 feet above datum, and are taken from a sheet of measured cross sections. Since the area at the base station is 2400, the values of m, column (5), are $2400/M$. The hydraulic radius, r, column (7), is taken as M/z, and the Chezy coefficients C, column (8), are from the Bazin formula, $C=87/(0.552+``m''/\sqrt{r})$, with $``m''=1.30$. The "m" in this formula has no relation to the ratio m in the tabulation. The corresponding values of p (equation 286) and of la/g (par. 373) are shown in columns (9) and (10).

398. The surface areas, U, at mean tide, between the velocity stations, and between the entrances and the adjacent velocity stations,

derived from a topographic survey made in 1928; and the consequent values of $I=aU/M_0=0.000, 1405\ U/2400$ are shown in the following tabulation.

From station	To station	Surface area, U (1,000's of square feet)	I	Storage station
5+000	7+500	325	0.019	6+250
7+500	19+500	9,907	.580	13+500
19+500	37+000	6,900	.404	28+250
37+000	53+000	5,300	.310	45+000
53+000	69+000	2,720	.159	61+000
69+000	77+000	1,847	.108	73+000

The storage stations are midway between the velocity stations.

399. *Initial computation of primary tides and currents.*—The computation of the tides and currents, by the procedure previously explained, is tabulated in figure 69 facing page 208. In columns (2) and (3) the coordinate amplitudes, A sin α and A cos α, of the tides at stations 5 and 77 are computed from the values of A and α at these stations, found in paragraphs 394 and 395. In the initial computation the values at the other stations are interpolated from equations (273) and (274). The values of B_0 sin β_0 and B_0 cos β_0 at station 37, columns (7) and (8), are determined from the head between stations 29 and 45. From columns (2) and (3) ,in this subsection:

$$H \sin H^\circ = -1.86-(-2.10) = +0.24$$

$$H \cos H^\circ = -0.37-(-0.75) = +0.38$$

$$H^\circ = 32^\circ 20',\ H=0.45,\ S=0.45/16{,}000=0.000{,}0281$$

$$\log P=0.28785,\ \log P/S=4.83914$$

$$\phi=17^\circ 10',\ \beta_0=H^\circ-\phi-90^\circ=-74^\circ 50'$$

$$B_0=1.90,\ B_0 \sin \beta_0=-1.83,\ B_0 \cos \beta_0=0.50$$

The initial computation is completed as explained in the first example, with due regard to the algebraic sign of the items. The minor discharge through the branch outlets from Biddles Point to the Delaware is neglected.

400. *Recomputation of primary tides and currents.*—In the recomputation, also shown in figure 69, the primary currents at the entrances, and the primary tides and currents at Biddles Point (station 19+500), and at Summit Bridge (station 51+200) are included. The underlined coordinate amplitudes of the tides in columns (2) and (3) are at the ends of the subsections, and are derived from the adjusted coordinate amplitudes of the heads found in the initial computation. Those at the other stations shown are interpolated. As in the previous example, no correction of the values of B_0 sin β_0 and B_0 cos β_0

at the base velocity station (station 37) is required. The coordinate increments to the current due to the flow through the branch channel extending from near Biddles Point to the Delaware are computed from the head in this channel. As this correction is small, the tides at the Delaware entrance to this branch may be taken as the same as those at the entrance to the main canal, at station 5. From the tabulated data, in this branch:

$$H \sin H^\circ = -2.29 - (-2.45) = +0.16$$

$$H \cos H^\circ = -1.30 - (-1.33) = +0.03$$

$$\tan H^\circ = 5.33 \qquad H^\circ = 79^\circ 20' \qquad H = 0.16$$

The length of the branch channel is 9,000 feet. Taking the bottom depth as 6 feet below datum, or 8.63 feet below mean tide, the bottom width as 50 feet, and the side slopes as 1 on 2, the area of the cross section is 580 square feet, the surface width is 85 feet and the hydraulic radius, r, is 6.8 feet. The corresponding value of C is 82. From this data:

$$\phi = 13^\circ 40' \qquad B = 0.965 \qquad \beta = -24^\circ 20'$$

$$m = 2400/580 = 4.14$$

Whence

$$(B/m) \sin \beta = -0.10 \qquad (B/m) \cos \beta = +0.21$$

These increments evidently are to be subtracted, together with the intervening storage, from the values of $(B/m) \sin \beta$ and $(B/m) \cos \beta$ at station 19+500 to give the values at station 7+500.

The values of $(B/m) \sin \beta$ and $(B/m) \cos \beta$ at station 51+200, enclosed in parentheses, are interpolated between those at the adjacent velocity stations. These values are disregarded in the summations by which the entries in columns (7) and (8) are derived.

The recomputed amplitudes and phases of the currents at the velocity stations are so close to those derived from the first computation that a recomputation of the heads is not made.

401. *Summary of results.*—The amplitudes and phases of the computed primary tides and currents, at the stations at which the actual tides and currents were observed, are shown in the following tabulation.

Computed primary tides and currents, Chesapeake and Delaware Canal

Station	$A \sin \alpha$	$A \cos \alpha$	$\tan \alpha$	α	A	B	β
5+000	−2.45	−1.33	----------	−118°20′	2.79	1.33	+19°30′
19+500	−2.29	−1.31	1.755	−119°40′	2.65	1.39	−43°10′
51+200	−1.84	−.66	2.78	−109°50′	1.95	2.39	−83°40′
77+000	−1.39	+.40	----------	−73°50′	1.45	2.90	−86°0′

Primary Tides and Currents.

Chesapeake and Delaware Canal, Nov. 27-28, 1928.

	1	2	3	4	5	6	7	8	9	10		11	12	13	14	15	16		17	18	19	20		21	22	23	24
																								Computed.		Adjusted.	
	Sta.	A sin α	A cos α	I	IA sin α	IA cos α	$\frac{B}{m}\sin\beta$	$\frac{B}{m}\cos\beta$	tan β (7)/(8)	β		$\frac{B}{m}$	m	B	p	tan φ p/B	φ		Ia/g	BIa/g	H (18)/sin φ	H° β+φ+90		H sin H°	H cos H°	H sin H°	H cos H°
1										°	′	Initial Computation.					°	′				°	′				
2	5+000	-2.45	-1.33																								
3	7+500						+0.37	+1.47	+0.252	14	10	1.51	0.78	1.18	0.537	0.455	24	30	0.022	0.026	0.063	128	40	+0.05	-0.04	+0.05	-0.04
4	13+500	-2.32	-1.13	0.580	-1.35	-0.66																					
5	19+500						-0.98	+0.81	-1.12	-50	30	1.27	1.00	1.27	0.585	0.461	24	50	0.083	0.105	0.250	64	20	+0.23	+0.11	+0.23	+0.11
6	28+250	-2.11	-0.77	0.404	-0.85	-0.31																					
7	29+000	-2.10	-0.75																								
8	37						-1.83	+0.50	-3.66	-74	50	1.90		1.90	0.585	0.308	17	10	0.070	0.133	0.450	32	20	+0.24	+0.38	+0.33	+0.38
9	45	-1.86	-0.37	0.310	-0.58	-0.11																					
10	53						-2.41	+0.39	-6.18	-80	50	2.44	1.00	2.44	0.683	0.280	15	40	0.070	0.171	0.635	24	50	+0.27	+0.58	+0.26	+0.58
11	61	-1.63	+0.02	0.159	-0.26	0																					
12	69						-2.67	+0.39	-6.84	-81	40	2.70	1.00	2.70	0.683	0.252	14	10	0.070	0.189	0.770	22	30	+0.30	+0.71	+0.29	+0.70
13	77	-1.39	+0.40																		Sums			+1.09	+1.74	+1.06	+1.73
14	(77)-(5)	+1.06	+1.73		-3.04	-1.08	-0.37	-1.47																+1.06	+1.73		
15							-3.04	-1.08																-0.03	-0.01		
16																											
17																											
18												Recomputation.															
19	5+000	-2.45	-1.33				+0.57	+1.61	+0.354	19	30	1.71	0.78	1.33													
20	6+250	-2.44	-1.34	0.019	-0.05	-0.03																					
21	7+500						+0.52	+1.58	+0.329	18	10	1.66	0.78	1.29													
22	10+000	-2.40	-1.37																								
23	13+500	-2.36	-1.35	0.580	-1.37	-0.78																					
24	Branch				-0.10	+0.31				-24	20	0.23	4.14	0.96													
25	19+500	-2.29	-1.31				-0.95	+1.01	-0.940	-43	10	1.39	1.00	1.39													
26	28+250	-2.18	-1.26	0.404	-0.88	-0.51																					
27	29+000	-2.17	-1.26																								
28	37						-1.83	+0.50	-3.66	-74	50	1.90	1.00	1.90													
29	45	-1.94	-0.88	0.310	-0.60	-0.27																					
30	51+200	-1.84	-0.66				(-2.36)	(+0.26)	-9.05	-83	40	2.39	1.00	2.39													
31	53						-2.43	+0.23	-10.56	-84	40	2.44	1.00	2.44													
32	61	-1.68	-0.30	0.159	-0.27	-0.05																					
33	69						-2.70	+0.18	-15.00	-86	10	2.73	1.00	2.73													
34	73	-1.46	+0.22	0.108	-0.16	+0.02																					
35	77	-1.39	+0.40				-2.86	+0.20	-14.30	-86	0	2.90	1.00	2.90													
36					-3.43	-1.41	-0.57	-1.61																			
37							-3.43	-1.41																			
38																											
39																											
40																											
41																											
42																											
43																											

FIGURE 69.

192750—40 (Face p. 208)

402. *Primary discharges and storage.*—Multiplying the area of the cross section at the entrances by the amplitude of the primary currents, it is found that, with the given tidal fluctuations, the primary discharge at Reedy Point reaches a maximum of 4,096 c. f. s.; and at the Chesapeake City entrance, 6,960 c. f. s. Evidently, the larger part of the filling and emptying of the canal is through the latter entrance.

403. *Effect of the flow through the canal upon the primary entrance tides.*—At Reedy Point the canal opens into the wide estuary of the Delaware, and the flow in and out of the canal obviously is insufficient to produce a measurable effect upon entrance tides. At the other entrance, at Chesapeake City, the discharge is into the comparatively restricted channel in Back Creek, whose area of cross section, in its upper part, is given as but 3,700 square feet. The discharge through the canal therefore, is sufficient to effect the currents and tides in this approach to the canal. The computation of the tides and currents in the canal has been made from the actual recorded elevations at Chesapeake City, after the canal was opened. If equally good records were available at the mouth of Back Creek, and physical data were at hand to determine the constants for the successive reaches in that approach, the computations profitably could have been extended to include this approach as a part of the canal prism.

DISTORTIONS OF THE PRIMARY CURRENTS

404. The distortions of the primary current in a short section of a tidal channel have been developed in paragraphs 260 to 276 of chapter V. In a long tidal canal, further distortions are introduced by the variation, with the rise and fall of the tide, in the area of the water surface between successive velocity stations, and in the area of the cross section of the water prism at these stations. The corrected velocities at any stations at which a determination is desired may be computed by a procedure which now will be described.

405. *Intervals.*—The corrected velocities are computed at selected intervals of time which, like those chosen for deriving the corrections in a short section of a tidal channel, should be parts of the component hour of the primary tides and currents. This component hour usually is the lunar hour of 1.035 mean solar hours. Intervals of one-half a lunar hour, or 1,863 mean solar seconds, ordinarily are sufficiently small to afford reliable results.

406. *Procedure.*—A first adjustment of the currents, which usually is sufficient for all practical purposes, may be accomplished through the following procedure:

(*a*) The primary tides at the ends of the subsections of the canal are adjusted to the selected representative tides at the entrances, if

these depart from the simple harmonic fluctuations of the primary entrance tides. The simplest adjustment, and one which appears as tenable as any other, is to assign the departures of the total head between the entrances to the primary heads in the subsections in proportion to the length of the subsection.

(*b*) The primary current in the subsection in which the amplitude, *B*, is the greatest is corrected to conform to the adjusted tides at the ends of the subsection, and for its other deformations, by the procedure described in chapter V. The currents in this subsection may be expected to have the largest influence upon the currents through the canal.

(*c*) The discharges at the velocity station of this base subsection are determined by multiplying the corrected velocity by the area of the cross section at this station at the given time.

(*d*) The currents at other stations are computed from those at this base station by a cubature of the adjusted tides through a modification of the process developed in chapter VI.

407. *Example.*—To illustrate the procedure, the corrected currents in the Chesapeake and Delaware Canal will be computed from the primary tides and currents derived in the preceding paragraphs, and entrance tides conforming to the observed tides. To curtail the computations, the small diurnal variation of the entrance tides is omitted and the tidal cycle is completed in 12 lunar hours. The representative tide at each entrance at zero hour (7 a. m., November 27, 1928) is taken as the mean of the recorded tides at 0 and 12 lunar hours; that at 0.5 lunar hour as the mean of the tides at 0.5 and 12.5 lunar hours; and so on. A minor adjustment at 11.5 hours produces fairly smooth repeating tide curves with a period of 12 lunar hours. The heights, above Delaware River datum, of the entrance tides so derived are as follows:

Selected entrance tides

Lunar hour	Station 5	Station 77	Lunar hour	Station 5	Station 77
0	0.90	3.35	6	3.35	2.50
.5	1.75	3.55	.5	2.80	2.10
1	2.55	3.85	7	2.25	1.75
.5	3.30	4.05	.5	1.65	1.45
2	4.00	4.15	8	1.15	1.35
.5	4.60	4.20	.5	.80	1.35
3	5.05	4.10	9	.40	1.35
.5	5.35	4.00	.5	.10	1.50
4	5.50	3.80	10	−.20	1.80
.5	5.30	3.52	.5	−.35	2.07
5	4.85	3.15	11	−.35	2.40
.5	4.20	2.90	.5	+.10	2.87

408. *Adjustment of primary tides at ends of subsections.*—The primary heads in the subsections are given by the equation:

$$h_s = H \cos (at + H^\circ)$$

in which H and $H°$ may be taken as the unadjusted values shown in columns (19) and (20) of figure 69, and *at* increases by 15° at each successive half lunar hour. Thus the surface head in the subsection between stations 61 and 77 at zero hour is:

$$0.770 \cos 22°30' = +0.71$$

and at 0.5 lunar hour it is:

$$0.770 \cos (15° + 22°30') = +0.61$$

and so on.

The tides at 0 hour are then adjusted to the selected entrance tides as follows:

(1) Station	(2) Primary head	(3) Factor	(4) Correction	(5) Adjusted head	(6) Adjusted tide
5					0.90
	−0.04	10/72	+0.05	+0.01	
10					.91
	+.11	19/72	.18	+.29	
29					1.20
	+.40	16/72	.15	+.55	
45					1.75
	+.58	16/72	.15	+.73	
61					2.48
	+.71	16/72	.16	+.87	
77					3.35
Sums	1.76	72/72	.69	2.45	

Total head, entrance tides	3.35−0.90=2.45
Primary tides	1.76
Total correction	.69

The primary heads in the subsections, column (2), total 1.76 feet, while the head between the selected entrance tides is 2.45 feet, giving a correction of 0.69 feet. The correction factors, column (3), are the lengths of the sections divided by the total length of the canal. The corrections derived by applying these factors to the total correction are shown in column (4), an odd hundredth being assigned to the subsection with the largest head. The adjusted tides, column (6), are found by successively adding the adjusted heads, column (5), to the tide at station 5, the initial entrance.

The tides at ends of the subsections at subsequent intervals of time are adjusted by a repetition of the process. The corrections generally are much smaller than those which, in this case, happen to occur at 0 hour.

409. The heights above Delaware River datum, of the adjusted tide at any other stations on the canal may be found, at half lunar hour intervals, by linear interpolation between the adjusted tides at the ends of the subsections. The adjusted and observed tides at Biddles Point (station 19+500) and at Summit Bridge (station 51+200) are

plotted in figure 70. The computed tides are seen to be in satisfactory concordance with the observed tides.

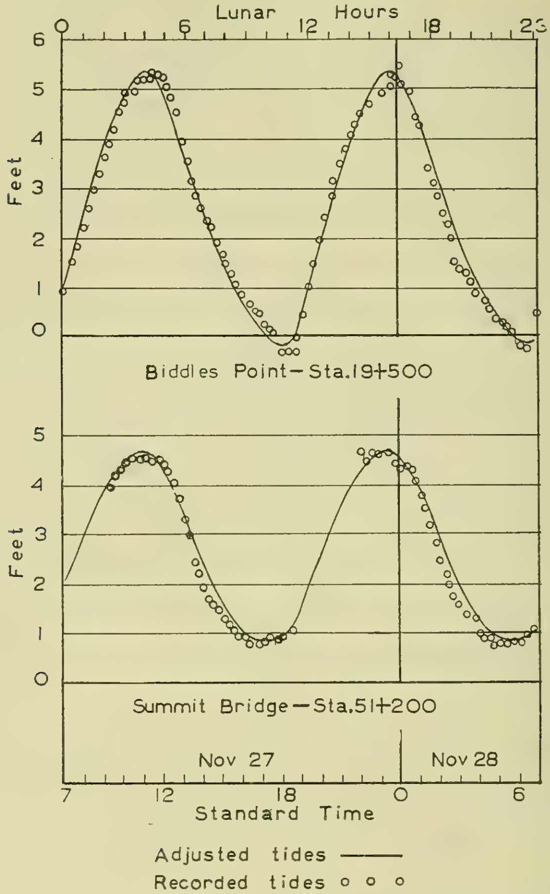

FIGURE 70.—Adjusted and observed tides, Chesapeake and Delaware Canal, November 27–28, 1928.

410. *Adjusted velocities and discharges at base velocity station.*—The primary currents are the largest in the subsection between stations 61 and 77. The corrected velocities at station 69, the velocity station of this subsection, are determined from the adjusted tides at stations 61 and 77, by the procedure described in paragraphs 260 to 276 of chapter V. The value of the hydraulic radius, r, as derived from the given cross section of this part of the canal, varies from 12.3 at 0 tide to 16.9 when the tidal height is 6.0 feet. The corresponding values of C, from the Bazin formula, with the coefficient used in the determination of the primary tides and currents, are 94.3 and 100, respectively. The

computations which are not here repeated, give the corrected velocities shown in column (3) of the following tabulation, from which the discharges are computed.

Discharges at station 69

(1) Lunar hour	(2) Primary current	(3) Corrected current	(4) Tide	(5) X	(6) Q
0	−2.73	−2.81	2.92	2,410	−6,770
.5	−2.59	−2.59	3.20	2,450	−6,350
1	−2.26	−2.31	3.58	2,520	−5,820
.5	−1.80	−2.02	3.87	2,570	−5,200
2	−1.20	−1.61	4.08	2,600	−4,190
.5	−.52	−1.03	4.23	2,640	−2,720
3	+.19	−.22	4.24	2,640	−580
.5	+.87	+1.05	4.22	2,630	+2,760
4	+1.56	+1.94	4.10	2,610	+5,060

The tidal heights at station 69, column (4), are the means of the adjusted heights at stations 61 and 77. From a sheet of typical cross sections, the area of the water prism at station 69 is found to be 1,900 square feet at 0 tide and 2,944 square feet at a 6-foot tide. The areas, X, of the cross section at the tidal heights shown in column (4) are taken off a straight line diagram and entered in column (5). These areas, multiplied by the velocities in column (3), give the discharges, Q, in cubic feet per second at half lunar hour intervals, shown in column (6). These, and subsequent computations, are by slide rule. The discharges through the rest of the 12 hour tidal cycle are computed in the same way.

411. *Discharges and velocities at other stations.*—The adjusted discharges and velocities at any other station on the canal are determined from the discharge at station 69 by the cubature of the adjusted tides through successive intervening reaches. As shown by equation (276), paragraph 368, the increase, ΔQ, in the discharge between two successive velocity stations is:

$$\Delta Q = -U \partial y / \partial t$$

in which U is the area of the water surface between the stations and $\partial y / \partial t$ is the rate at which the average tidal height between the stations is increasing with the time. For sufficiently small increments of time, Δt, this equation may be written:

$$Q = -U \Delta y / \Delta t$$

in which U is the area of the water surface at a given time, Δt is the selected time interval, and Δy may be taken as the mean increase in the average tide between the stations during the preceding and following intervals. In the present case, Δt is the half lunar hour of 1,863 seconds.

412. The cubature between stations 69 and Summit Bridge, station 51+200, takes the following form:

Cubature—stations 69 to 51+200

Lunar hour	Tides			Increments		$U/\Delta t$	ΔQ	Q		Area X	V
	Station 69	Station 51+200	Mean	Prior	Δy			Station 69	Station 51+200		
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
0	2.92	2.05	2.48	+0.54	+0.46	1,795	−830	−6,770	−5,940	2,250	−2.64
.5	3.20	2.51	2.85	+.37	+.42	1,815	−760	−6,350	−5,590	2,330	−2.39
1	3.58	3.04	3.31	+.46	+.42	1,840	−770	−5,820	−5,050	2,430	−2.00
.5	3.87	3.52	3.69	+.38	+.35	1,855	−650	−5,200	−4,550	2,520	−1.80
2	4.08	3.93	4.00	+.31	+.29	1,870	−540	−4,190	−3,650	2,580	−1.41
.5	4.23	4.29	4.26	+.26	+.19	1,880	−360	−2,720	−2,360	2,640	−.90
3	4.24	4.51	4.37	+.11	+.09	1,890	−170	−580	−410	2,680	−.15
.5	4.22	4.66	4.44	+.07	+.02	1,890	−40	+2,760	+2,800	2,710	+1.03
4	4.10	4.69	4.40	−.04	−.14	1,890	+270	+5,060	+4,790	2,720	+1.76
.5	3.85	4.50	4.17	−.23							

The tides at station 69, column (2), are those shown in column (5) of the preceding tabulation. Those at station 51+200, column (3), are derived by linear interpolation between the adjusted tides at stations 45 and 61. The mean tide in the reach from station 51+200 to station 69 is shown in column (4). The increase in the mean tide in the preceding interval is entered in column (5), the entry at 0 hour being repeated from that for 12 hours (not shown). The mean of the entries on the half hour and on the succeeding half hour, in column (5), gives the mean rise during the preceding and following intervals, and is the value of Δy, column (6). The area, U, of the water surface between stations 69 and 51+200, from topographical maps of the canal, is 3,128,000 square feet at zero tide, and 3,662,000 square feet at a 6.0 foot tide. Dividing by $\Delta t=1863$, the value of $U/\Delta t$ at 0 tide is 1,679, and at 6-foot tide, 1,966. The values of $U/\Delta t$ at the mean tidal heights shown in column (4) are taken off a straight line diagram and entered in column (7). The values of ΔQ, column (8), are then the products of the entries in columns (6) and (7), with the sign reversed. The discharges at station 69, previously found, are entered in column (9). Since the cubature is in the negative direction, the values of ΔQ are subtracted therefrom, algebraically, to give the discharges at station 51+200, column (10). The typical cross sections of the canal show the same section at station 51+200 as at station 69, and the areas X of the cross section at the latter station, at the tidal heights shown in column (3) are taken from the diagram previously prepared. The quotient of the entries in columns (10) and (11) then

gives the velocities at station 51+200, in column (12). The computation for the rest of the 12-hour cycle is in the same form.

413. The velocities at the entrance to the canal at Chesapeake City Bridge, station 77, are similarly derived by cubature from station 69. The discharges at station 37 are derived from those at 51+200, and thence successively the discharges and velocities at station 19+500 (Biddles Point) and at station 5 (Reedy Point). The flow through the branch canal which makes off from near Biddles Point is too small to warrant the labor involved in including it in the adjustment. The surface areas, U, and the areas of the cross sections, used for these computations, are as follows:

Surface areas, U (square feet)

Reach	*0 tide*	*6.0-foot tide*
Stations 69 to 77—prism	1, 406, 000	1, 646, 000
Back Creek	583, 000	4, 294, 000
Total	1, 989, 000	5, 940, 000
Stations:		
51+200 to 69	3, 128, 000	3, 662, 000
37 to 51+200	3, 505, 000	7, 699, 000
19+500 to 37	4, 294, 000	9, 447, 000
5 to 19+500	8, 208, 000	13, 568, 000

Cross sections, X (square feet)

Stations:	*0 tide*	*6.0-foot tide*
From 45 to 77	1, 900	2, 944
19+500	1, 936	3, 065
5	2, 428	4, 060

414. *Comparison with observed velocities.*—The computed primary currents and adjusted currents, at Reedy Point (station 5) and at Biddles Point (station 19+500) and the mean velocities from meter measurements at these stations on November 27–28, 1928, are shown in figure 71, page 216. It may be seen that the adjustments produce large distortions of the primary velocity curves at these stations. While the recorded velocities are somewhat erratic, the adjusted velocities conform fairly well to the observations. The computed and observed currents at Summit Bridge (station 51+200) and at Chesapeake City Bridge (station 77) are shown in figure 72, page 217. The recorded velocities at these stations are much more consistent, and with some minor variations the adjusted currents resulting from the computations are in close accordance therewith. Although the constants and data used in the computations were selected without regard to the observed currents, the agreement is perhaps closer than could ordinarily be expected.

415. *Second approximation.*—The corrections applied to the velocities in the subsections of the canal must change, to some extent, the distribution of the total surface head between the subsections. A second adjustment may be made by computing the surface heads produced by the corrected velocities in each subsection of the canal, at

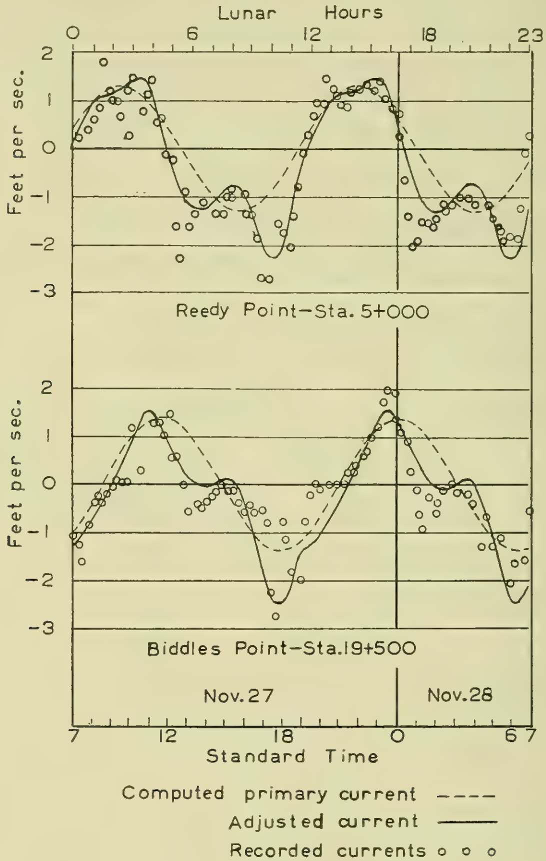

FIGURE 71.—Computed and observed currents, Chesapeake and Delaware Canal, November 27–28, 1928

the adopted half lunar hour intervals. These heads are determined from the equation:

$$h_s+h_v+h_a+h_f=0$$

by computing the values of h_v, h_a, and h_f from the corrected velocities. The surface heads so computed may then be adjusted to the heads established by the selected entrance tides, corrected tidal heights at the ends of the subsections derived therefrom, and a second determination of the velocities made from the corrected tides. The procedure is too laborious to be warranted in any ordinary case. The results of its application to the tides and currents in the canal chosen for the first example, in which the entrance tides were taken with a simple harmonic

fluctuation, are shown in figure 73, page 218; and the instantaneous profiles derived therefrom in figure 74, page 219. In these figures the tides are referred to a datum 10 feet below mean sea level. It may be noted that the changes produced in the tides by the adjust-

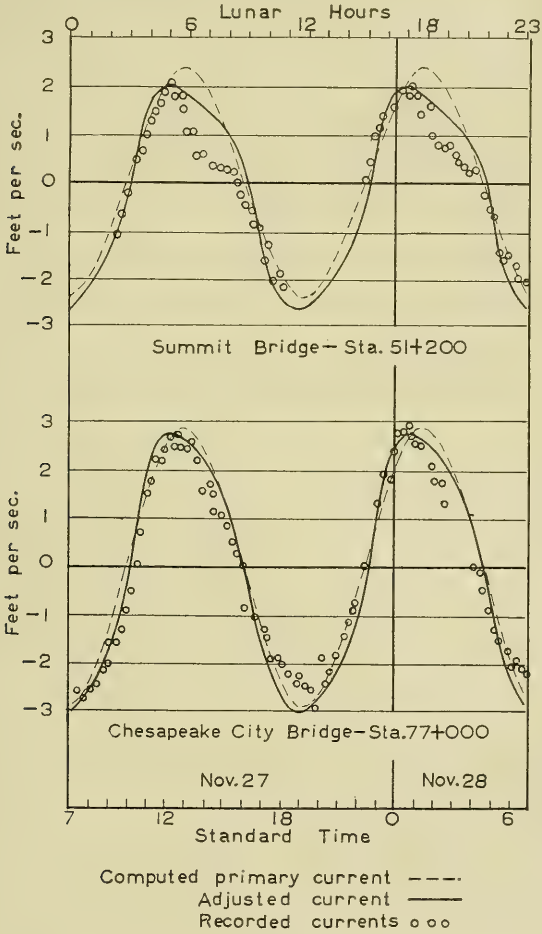

FIGURE 72.—Computed and observed currents, Chesapeake and Delaware Canal, November 27–28, 1928

ment are very small. The currents derived from the second adjustment do not differ materially from those derived in the first. The weaker currents at the initial entrance show a considerable distortion. A plot of the distorted discharges at the two entrances, not here shown, develops no material departures from the relation between the storage and discharges due the primary currents in the canal, discussed in paragraph 388.

416. *Preponderance of flow in a connecting canal.*—The flow in the two directions through a connecting canal ordinarily is at different stages of the tide. Because of the consequent difference in the mean

areas of the cross section of the water prism while the flow is in the opposite directions, and because of the distortions of the currents, the total volume of flow in one direction may be expected to differ from that in the other. This preponderance of flow may be estimated by computing the arithmetical mean of the adjusted discharges at the given intervals.

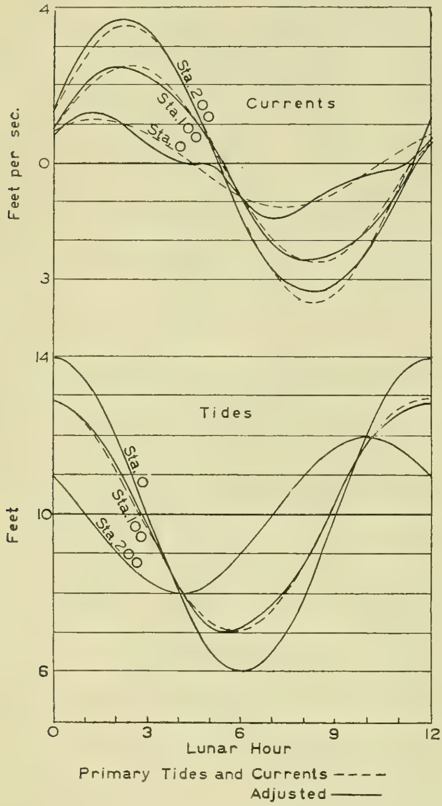

FIGURE 73.—Primary and adjusted currents in first example

417. In the canal selected in the first example the algebraic mean of the finally computed discharges at the initial entrance is −275 c. f. s. and at the further entrance it is −158 c. f. s. A closer adjustment would be necessary to remove the discrepancy between these two figures. The maximum discharges at these entrances are 22,220 c. f. s. and 50,700 c. f. s., respectively. These figures show that in this case the total volume of flow is nearly the same in both directions, but indicate a slight preponderance of flow toward the initial entrance, where the tidal range is the greater.

418. In the Chesapeake and Delaware sea-level barge canal, taken as the second example, the adjusted discharges produced by entrance tides on the day selected show a preponderance of flow averaging −441 c. f. s. at the initial entrance, station 5, and of −376 c. f. s. at the further entrance, station 77. These figures indicate an average net discharge during the day of about 400 c. f. s. through the canal in the negative direction, from Chesapeake Bay into Delaware River. This preponderance of flow may be attributed to the higher mean tide elevation in the head of Chesapeake Bay.

419. It is not difficult to see that in a comparatively short canal, with a wide difference in the tidal range at the two entrances, the tidal elevations and the surface heads through the canal are dominated by the tide at the entrance having the larger tidal range; and because of the greater cross section and less resistance to flow at the higher tidal

stages at that entrance, more water will flow through the canal from that entrance than will flow back at low tide when the direction of the flow is reversed; provided at least that no adverse constant component of the head is produced by a difference in the elevation of mean tide at the two entrances. The more intricate tides and currents in a longer canal and differences in the elevation of mean tide at the entrances may produce a preponderance of flow which is not necessarily from the entrance having the larger tidal range.

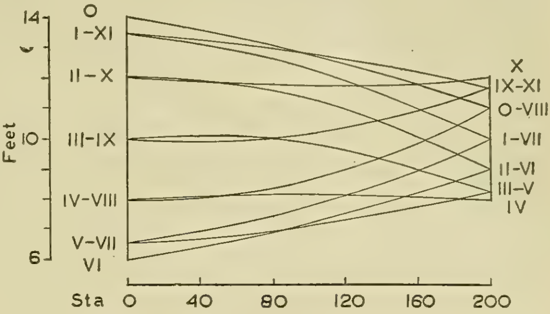

FIGURE 74.—Instantaneous profiles (adjusted).

CLOSED CANALS

420. A computation of the currents and tides in a projected closed canal seldom is necessary, as usually it may be taken for granted that the currents in such a canal will not be troublesome; but should the occasion arise, the primary tides and currents may be computed by a procedure paralleling that applied in the preceding paragraphs to connecting canals. Aside from a practical application, the development of the effect of frictional resistance upon the primary tides and currents in a long closed canal of uniform dimensions will cast some light upon the characteristics of tidal flow in closed channels in general.

421. *Computation for closed canals of moderate length.*—If a projected canal is so short that the instantaneous profiles will not depart widely from horizontal lines, the computations may be started by determining the currents that would be produced in successive subsections of the canal if the primary tides in each subsection had the same amplitude and phase as at the entrance. The surface heads in the subsections are then computed, corrected tides derived therefrom, the currents recomputed, and the computations repeated until further corrections become negligible.

422. Since the discharge at the head of the canal is zero, the discharge, Q, at a velocity station at the middle of any subsection is, from equation (278):

$$Q = MB \sin(at+\beta) = \Sigma aUA \sin(at+\alpha) \tag{293}$$

in which M is the area of the cross section at the velocity station, B the amplitude and β the initial phase of the primary current at the station; and $\Sigma aUA \sin(at+\alpha)$ is the summation, from the head of the

canal to the velocity station, of the products of the surface areas, U, between the successive velocity stations, at mean tide, and aA sin $(at+\alpha)$, the rate of increase of the tide at the storage stations midway between the velocity stations.

Designating the area of any typical cross section of the canal as M_0, and placing, as in equations (281) and (282):

$$M_0/M=m$$

$$aU/M_0=I$$

equation (293) may be written:

$$B \sin (at+\beta)=m\Sigma IA \sin (at+\alpha)$$

whence:

$$(B/m) \sin \beta=\Sigma IA \sin \alpha \qquad (294)$$

$$(B/m) \cos \beta=\Sigma IA \cos \alpha. \qquad (295)$$

If the canal is of uniform dimensions, and the subsections of equal length, $M_0=M$, $1/m=1$, and $I=a\Delta x/D$.

The component currents in the subsections are computed from equations (294) and (295), and the resulting surface heads from equations (288), (289), and (290).

423. *Example.*—The computations may be illustrated by applying them to a closed canal of uniform cross section, 60,000 feet (11 miles) in length, with a mean depth of 16 feet at midtide, when the tidal fluctuation in the entrance has a range of 6 feet, and the speed of the M_2 component, 0.0001405 radians per second. The origin of distances is at the head of the canal, and the origin of time at a high water at the entrance. The canal will be divided into three subsections, each 20,000 feet in length, as shown in figure 75.

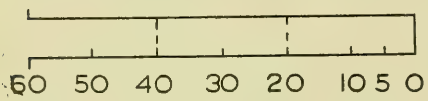

FIGURE 75.—Storage and velocity stations (stations of 1,000 feet).

Station 0 is at the head of the canal. The velocity stations, at the middle of the subsections, are at stations 10, 30, and 50. The storage stations are at stations 5, 20, and 40. As the currents near the head of the canal are extremely small, the surface head in the quarter section between stations 0 and 5 is always negligible, and the components of the tide at station 5 may be taken as those at station 0.

424. *Coefficients.*—An appropriate value of Chezy coefficient, C, is 100. The storage coefficient, I, at all storage stations, except for the half section at the head of the canal, is:

$$I=a\Delta x/D=0.0001405\times 20{,}000/16=0.176.$$

For the half section at the head of the canal, $I=0.088$.

Since the canal is of uniform dimensions, $m=1$. The coefficients for the determination of the subsection heads (par. 373) are:

$$p=0.000,005,148\ C^2r=0.823$$

$$la/g=0.000,004,37\times 20,000=0.0874.$$

425. *First computation of the primary currents and heads.*—The computations are started by taking the equation of the tide throughout the canal as:

$$y=3 \cos at.$$

At all storage stations therefore:

$$A \cos \alpha=3 \qquad A \sin \alpha=0.$$

The computations are conveniently made in the form previously used for connecting canals, and are shown on figure 76, facing page 222. The value of $(B/m) \cos \beta$ at station 10, is the storage increment for the half subsection, 0 to 10; that at station 30 is obtained by adding the storage increment between stations 10 and 30, and so on. The subsection velocities and heads are then computed, but since the tide at the entrance to a closed canal is alone fixed, the computed coordinate heads are not subject to adjustment.

426. *Recomputation of currents and heads.*—The currents and heads are next recomputed as shown in figure 76 from the tides established by the heads determined in the initial computation. The component tides, $A \sin \alpha$ and $A \cos \alpha$, at stations 40, 20, and 0 are obtained by successively subtracting, algebraically, the component heads, $H \sin H°$ and $H \cos H°$, found in the first computation, from the component tides at station 60.

In the final computation the current at the entrance, station 60, is determined by adding to the component currents at station 50, the storage increments from stations 50 to 60. The component tides at the storage station, station 55, are interpolated.

427. *Results of computation.*—The amplitudes and initial phases of the tides at the ends of the subsections, derived from the final computation, are:

Station:	A	α
60	3.00	0
40	3.12	−3°50′
20	3.19	−5°10′
0	3.21	−5°20′

The tidal range therefore increases from 6.0 feet at the entrance to 6.42 feet at the head of the canal. High water at the head of the canal is 5°.33/28°.98=0.18 hours=11 minutes later than at the entrance. The strength of the current at all sections is nearly at midtide, and decreases from 1.66 feet per second at the entrance to zero at the head of the canal. The currents are so weak that the tides and currents approach the condition of frictionless flow.

428. *Computations for a longer canal.*—The procedure which has been described is applicable only to a comparatively short canal. As

the length increases, the successive approximations converge more slowly, and after a certain length is reached, run completely wild. To compute the primary tides and currents in a long closed canal, the amplitudes and initial phases of the currents produced by tides of successive amplitudes at a station at a moderate distance from the head of the canal may be determined by the method that has been described. The tides and currents set up when another section is added are derived therefrom. By continuing the process, the primary tides and currents in a closed canal of any length may be computed.

429. *Example.*—The primary currents produced by tides of successive amplitudes at the entrance to a canal 60,000 feet in length, and 16 feet in mean depth, determined by the same procedure as that set forth in figure 76, are:

At entrance (station 60)				*At head (station 0)*	
TIDE		CURRENT		TIDE	
A	α	B	β	A	α
3.5	0	1.94	$-4°50'$	3.73	$-6°10'$
3.0	0	1.66	$-4°$	3.21	$-5°20'$
2.5	0	1.40	$-3°20'$	2.70	$-4°30'$

The computation from this data of the primary currents at the entrance to a canal 80,000 feet long and of the same mean depth, when the tide at the entrance has an amplitude of 3 feet, is shown at the bottom of figure 76. For the initial computation the tide at station 60 is taken as the same as at station 40 of the 60,000 foot canal, the amplitude of which is 3.12 feet and the initial phase is $-3°50'$. The corresponding amplitude of the current is, by interpolation from the tabulated data, 1.73 feet per second, and its initial phase, for a zero phase of the tide at station 60 is $-4°10'$. Since the phase of the tide at station 60 is $-3°50'$, the phase of the current at this station is $-4°10'-3°50' \doteq -8°$. The coordinate amplitudes of the velocity at station 60 are then:

$$1.73 \sin(-8°) = -0.240 \qquad 1.73 \cos(-8°) = 1.710.$$

The current at station 70 is derived by adding the velocity increment, stations 60 to 70, determined by the tide at station 65, and the resulting coordinate heads, stations 60 to 80 derived therefrom. The recomputation from the corrected tides at station 60, develops heads in satisfactory agreement with those first found. In the final computation, the current at station 80 is determined by adding to the corrected velocity components at station 60 the storage due to the tide at station 70. The current at the entrance to the 80,000 foot canal is found to have an amplitude of 2.27 feet per second, and an initial phase of $-10°30'$.

The final determination of the amplitude of the tide at station 60 of the 80,000-foot canal is 3.13 feet and its phase is $-8°20'$. The

Primary Tides and Currents.

Closed canal, 60,000 feet long, mean depth 16 feet

	1	2	3	4	5	6	7	8	9	10		11	12	13	14	15	16		17	18	19	20		21	22	23	24
	Sta.	$A \sin\alpha$	$A \cos\alpha$	I	$IA \sin\alpha$	$IA \cos\alpha$	$\frac{B}{m}\sin\beta$	$\frac{B}{m}\cos\beta$	$\tan\beta$ (7)/(8)	β		$\frac{B}{m}$	m	B	p	$\tan\phi$ p/B	ϕ		Ia/g	BIa/g	H (18)/$\sin\phi$	$H°$ $\beta+\phi+90$		Computed $H \sin H°$	Computed $H \cos H°$	Adjusted $H \sin H°$	Adjusted $H \cos H°$
1										°	′	Initial computation.					°	′				°	′				
2	0-5	0	3.00	0.088	0	0.264																					
3	10						0	0.264	0	0				0.26	0.823	3.164	72	30	0.0874	0.023	0.024	162	30	0.01	-0.02		
4	20	0	3.00	0.176	0	0.528																					
5	30						0	0.792	0	0				0.79		1.042	46	10		0.069	0.096	136	10	0.07	-0.07		
6	40	0	3.00	0.176	0	0.528																					
7	50					1.320	0	1.320	0	0				1.32		0.623	32	0		0.115	0.216	122	0	0.18	-0.11		
8	60	0	3.00																					0.26	-0.20		
9												Recomputation.															
10	0-5	-0.26	3.20	0.088	-0.023	0.282																					
11	10						-0.023	0.282	-0.082	-4	40			0.28	0.823	2.940	71	10	0.0874	0.024	0.025	156	30	0.01	-0.02		
12	20	-0.25	3.18	0.176	-0.044	0.560																					
13	30						-0.067	0.842	-0.079	-4	30			0.84		0.980	44	30		0.073	0.105	130	0	0.08	-0.07		
14	40	-0.18	3.11	0.176	-0.032	0.547																					
15	50				-0.099	1.389	-0.099	1.389	-0.071	-4	00			1.39		0.592	30	40		0.121	0.238	116	40	0.21	-0.11		
16	60	0	3.00																					0.30	-0.20		
17												Final computation.															
18	0-5	-0.30	3.20	0.088	-0.026	0.282																					
19	10						-0.026	0.282	-0.092	-5	20			0.28	0.823	2.940	71	10	0.0874	0.024	0.025	155	50	0.01	-0.02		
20	20	-0.29	3.18	0.176	-0.051	0.560																					
21	30						-0.077	0.842	-0.091	-5	10			0.84		0.980	44	30		0.073	0.105	129	20	0.08	-0.07		
22	40	-0.21	3.11	0.176	-0.037	0.547																					
23	50						-0.114	1.389	-0.082	-4	40			1.39		0.592	30	40		0.121	0.238	116	0	0.21	-0.10		
24	55	-0.05	3.03	0.088	-0.004	0.267																		0.30	-0.19		
25	60	0	3.00		-0.118	1.656	-0.118	1.656	-0.071	-4	00			1.66													
26												Canal 80,000 feet long.															
27												Initial computation.															
28	60	$\tan\alpha = -0.0675$, $\alpha = -3°50'$, $A = 3.12$, $B = 1.73$, $\beta = -4°10' - 3°50' = -8°$																									
29	60	-0.21	3.11				-0.240	1.710																			
30	65	-0.16	3.08	0.088	-0.014	0.271																					
31	70						-0.254	1.981	-0.128	-7	20			2.00	0.823	0.411	22	20	0.0874	0.175	0.461	105	0	0.45	-0.12		
32	80	0	3.00																								
33												Recomputation.															
34	60	$\tan\alpha = -0.144$, $\alpha = -8°20'$, $A = 3.15$, $B = 1.74$, $\beta = -4°10' - 8°20' = -12°30'$																									
35	60	-0.45	3.12				-0.376	1.700																			
36	65	-0.34	3.09	0.088	-0.030	0.272																					
37	70						-0.406	1.972	-0.206	-11	40			2.01	0.823	0.409	22	20	0.0874	0.176	0.464	100	40	0.45	-0.09		
38	80	0	3.00																								
39												Final computation.															
40	60	$\tan\alpha = -0.1455$, $\alpha = -8°20'$, $A = 3.13$, $B = 1.73$, $\beta = -4°10' - 8°20' = -12°30'$																									
41	60	-0.45	3.09				-0.375	1.700																			
42	70	-0.225	3.05	0.176	-0.040	0.538																					
43	80	0	3.00				-0.415	2.228	-0.186	-10	30			2.27													

FIGURE 76.

corresponding amplitude of the tide at the head of the canal is, by interpolation from the tabulated data, 3.34 feet; and its initial phase is $-5°30'-8°20'=-13°50'$. The primary tides and currents at any other station on an 80,000 foot canal could be determined in a similar manner by establishing their relation to the amplitude of the tide at station 60.

Perhaps a better method for computing the tides and currents in a very long closed canal is to determine those that would be produced in the successive subsections by tides of several amplitudes at the head of the canal. If the subsections are 20 stations in length, the current produced at station 10 by a tide of given amplitude at the head of the canal is derived from the velocity increments from the tide at station 5, and the coordinate heads, tides and currents at station 20 computed therefrom. The coordinate amplitudes of the tide at station 25 can then be set forward with fair assurance and the currents at station 30 determined by adding the velocity increments due to the tide at station 25 to the coordinate currents at station 20; and so on to the station at the entrance. The amplitudes and phases of the tide and current at any station on the canal can then be plotted against the several computed amplitudes of the tide at the entrance, and those corresponding to an entrance tide of a given amplitude taken off these diagrams.

430. *Characteristics of the tides and currents in a long closed canal of uniform cross section.*—The primary tides and currents in a canal 140,000 feet (about 26.5 miles) in length, of uniform cross section, 16 feet in mean depth, produced by an entrance tide of 3 foot amplitude, as computed by the step by step process just outlined, are as follows:

Primary tides and currents in closed canal 140,000 feet long, with a mean depth of 16 feet at mean tide

[C=100]

Station (1,000 feet)	Tide		Current	
	Amplitude A	Phase	Amplitude B	Phase
140	3.0	0	3.2	−44°30′
120	2.6	−18°30′	2.8	−50°30′
100	2.5	−36°10′	2.4	−54°50′
80	2.6	−48°20′	2.0	−57°50′
60	2.7	−55°40′	1.5	−59°20′
40	2.8	−59°20′	1.0	−60°20′
20	2.86	−60°40′	.5	−60°40′
0	2.9	−60°50′	0	----------

The angular lag, ϕ, of the current increases from 15° in the entrance subsection, station 120 to 140, to 90° at the head of the canal. Toward the entrance the flow becomes largely frictional, while near the head it is essentially frictionless.

The range of the tide decreases from 6 feet at the entrance to 5.0 feet at a point 40,000 feet (about 8 miles) up the canal, and thence increases to 5.8 feet at the head of the canal. High water at the head of the canal is $60^\circ.8/28.98=2.1$ mean solar hours later than at the entrance. The rate at which the tide progresses up the canal is far from uniform. In the first subsection next the entrance it progresses at the rate of 8.2 feet per second, while in the upper 40,000 feet it progresses at an average rate of over 200 feet per second. The rate of advance of a progressive wave in a canal of the given depth would be $\sqrt{16g}=22.7$ feet per second.

The instantaneous profiles in the canal at successive lunar hours are shown in figure 77.

In an even longer canal of the same depth the rate of progress of the tide is found to decrease slowly from the entrance for some distance up the channel and thence increase rapidly toward the head.

431. The primary current at the entrance reaches a maximum of 3.2 feet per second, and the strength of the positive current occurs

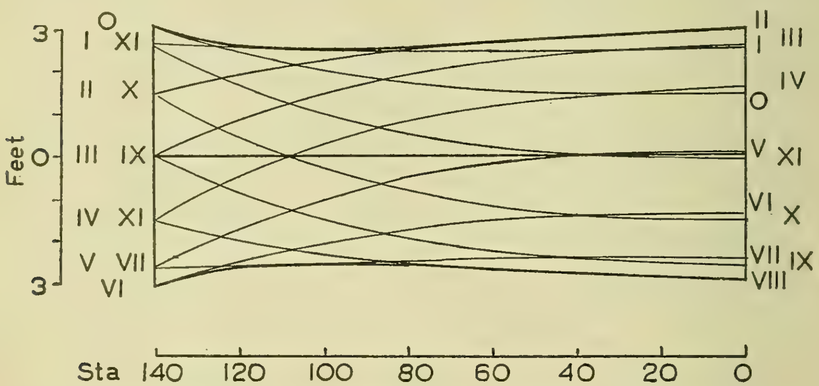

FIGURE 77.—Instantaneous profiles in closed canal 140,000 feet long and 16 feet mean depth.

1.5 mean solar hours before high water, or 1.6 hours after midtide. The strength of the current decreases nearly uniformly to the head of the canal. Near the head of the canal the strength of the positive current is 30 minutes later than at the entrance, and occurs nearly at midtide at the head of the canal.

Considerable deformations of the primary currents are to be expected in so long and shallow canal; but the many successive approximations necessary to bring the deformations of the tides and currents into concordance would render their computation excessively laborious.

432. *Computation for canal of varying cross section.*—If the surface width and the mean depth of a closed canal are not the same throughout, the values of the coefficients, I, p, and la/g are determined for each subsection, and the form of the computations is modified in the

same manner as those of a connecting canal of varying cross section, illustrated in the second example (pars. 392–401).

MIDSTREAM CURRENTS

433. The computations developed in this chapter should afford substantially as reliable an indication of the mean velocity at a given cross section of a tidal canal as is to be expected of a computation of the mean velocity set up by steady flow with a constant head. In both cases, the reliability of the results depends upon the completeness of the data on the actual widths and depths in the canal, and on the selection of the coefficient of roughness. The procurement of the data for the computations generally entails much more effort than do the computations themselves.

434. It should be recollected that the currents which will be encountered in the navigation of the canal are those in midstream and that their velocity considerably exceeds the mean velocity in the cross section. An analysis of the detailed meter measurements made in the Cape Cod Canal in 1915, when its designed depth was 25 feet at low water and its bottom width 100 feet, shows that the average velocity in a vertical section at the middle of the canal was 25 percent in excess of the average velocity in the entire cross section. While the ratio of the midstream velocity to the mean velocity must depend upon the contour of the bed of the channel in the vicinity of the cross section, available data indicates that in general the strength of the midstream current in a channel of regular dimensions should be taken as 1.3 times the mean strength.

435. The midstream current also turns later and reaches its maximum velocity after the mean current in the cross section. In a canal of regular section this difference in timing usually does not exceed a few minutes. In a wide natural channel differences of half an hour or more in the time of the turning of the current near the shore and at midstream are quite common (Manual of Current Observations, Special Publication 215, U. S. Coast and Geodetic Survey, 1938).

CHAPTER IX

TIDES AND CURRENTS IN ESTUARIES AND INLETS

436. *Definition of estuary.*—The reversing currents produced by the filling and emptying of the tidal prism of a river that enters a tidal sea, generally dominate the river flow for a considerable distance up the stream. This part of the river usually is funnel shaped, flaring towards the entrance. A river mouth of such a shape is called an *estuary*. The term may be applied as well to any tidal channel of similar shape, even if it does receive any considerable inflow from the uplands.

437. *Characteristic tides and currents in an estuary.*—In a typical estuary, the currents often have nearly the same strength and tides nearly the same range at all cross sections; except in the upstream reaches where the tidal flow merges into steady flow. The rate at which high water and low water, and the strength and turn of the current, advance up an estuary is often so close to $\sqrt{gD}$, the rate of advance of a frictionless progressive wave in a channel of uniform dimensions (par. 339), that this is commonly regarded as the normal rate of progress of the tide.

438. *The ideal estuary.*—In the preceding chapter it was shown that the currents in a long closed canal of uniform cross section diminish from the entrance to the head of the canal and the rate at which the tide advances up the canal increases toward the head. The uniformity of the currents in an estuary, and of the rate of advance of the tide, evidently is due to its shape. It is of interest to determine the special

shape that a closed tidal channel of constant depth must have, in order that a simple harmonic fluctuation of the tide at the entrance will produce throughout the channel primary tides of constant range and primary currents of uniform strength. In the lack of a better term, a channel of this shape may be called an *ideal* estuary.

439. *Derivation of the form of an ideal estuary.*—Taking the origin of distances at the entrance from the sea, the positive direction upstream, and the origin of time at a high water at the entrance, let:

- D, be the constant mean depth of the channel at mean tide.
- z, its width at a point distant x from the origin.
- r, its constant hydraulic radius at mean tide.
- C, the applicable Chezy coefficient; also taken as constant.
- A, the constant amplitude of the primary tide.
- B, the constant amplitude of the primary current.
- a, the speed of the primary tides and currents.
- S, the amplitude of the surface slope at a point distant x from the origin.
- $H°$, the initial phase of the slope at the same point.
- ϕ, the angular lag of the current.

The relations established in paragraph 373 show that if B, r, and C have constant values in a given channel, the values of ϕ and S also are constant throughout the channel.

The equation of the tide at the entrance is:

$$y = A \cos at.$$

Since the tide at a station within the entrance occurs at a later time, its equation is in the form:

$$y = A \cos (at - \zeta) \tag{296}$$

in which ζ (zeta) is a positive angle which varies with x.

The surface slope at a point distant x from the origin, and at the time t, is then:

$$\begin{aligned} S \cos (at + H°) = \partial y/\partial x &= A \sin (at - \zeta)\, \partial\zeta/\partial x \\ &= A \cos (at - \zeta - \pi/2)\, \partial\zeta/\partial x. \end{aligned} \tag{297}$$

Since equation (297) is identically true:

$$S = A\,\partial\zeta/\partial x \tag{298}$$

$$H° = -\zeta - \pi/2. \tag{299}$$

From equation (298)

$$\partial\zeta = (S/A)\,\partial x.$$

The integration of which gives, since $\zeta=0$ when $x=0$:

$$\zeta=(S/A)x. \tag{300}$$

It will be convenient to place:

$$S/A=n. \tag{301}$$

So that:

$$\zeta=nx. \tag{302}$$

The equation of the current at any cross section of the channel may be written:

$$v=B\sin(at+\beta)$$

in which, from equation (150):

$$\beta=H^\circ-\phi-\pi/2$$

From equations (299) and (302):

$$\beta=-nx-\phi-\pi.$$

The equations of the tide and current in an ideal estuary are then:

$$y=A\cos(at-nx) \tag{303}$$
$$v=B\sin(at-nx-\phi-\pi)=-B\sin(at-nx-\phi). \tag{304}$$

These equations show that the tides and currents progress up an ideal estuary at the constant rate of a/n.

440. The differential equation of the primary current has been derived in equation (142), paragraph 243:

$$\partial y/\partial x+(1/g)\partial v/\partial t+(8/3\pi)Bv/C^2r=0$$

Substituting the differential coefficients and the expression for v obtained from equations (303) and (304):

$$An\sin(at-nx)-(aB/g)\cos(at-nx-\phi)$$
$$-(8/3\pi)(B^2/C^2r)\sin(at-nx-\phi)=0 \tag{305}$$

By placing $at-nx=0$, the equation of condition is derived:

$$-(aB/g)\cos\phi+(8/3\pi)(B^2/C^2r)\sin\phi=0 \tag{306}$$

and by placing $at-nx=\pi/2$:

$$An-(aB/g)\sin\phi-(8/3\pi)(B^2/C^2r)\cos\phi=0 \tag{307}$$

Multiplying equation (306) by $\cos\phi$ and equation (307) by $\sin\phi$ and adding:

$$An\sin\phi - aB/g = 0 \tag{308}$$

441. The equation of continuity is (equation 183):

$$\partial(vzD)/\partial x + z\partial y/\partial t = 0$$

When the depth, D, is constant, this equation becomes:

$$Dv\partial z/\partial x + Dz\partial v/\partial x + z\partial y/\partial t = 0 \tag{309}$$

Substituting the differential coefficients and the expression for v, from equations (303) and (304):

$$-DB\sin(at-nx-\phi)\partial z/\partial x + nDBz\cos(at-nx-\phi)$$
$$-aAz\sin(at-nx) = 0 \tag{310}$$

The equations of condition, derived by placing $at-nx=0$ and $at-nx=\pi/2$, are:

$$DB\sin\phi\partial z/\partial x + nDBz\cos\phi = 0 \tag{311}$$
$$-DB\cos\phi\partial z/\partial x + nDBz\sin\phi - aAz = 0 \tag{312}$$

Multiplying equation (311) by $\cos\phi$ and equation (312) by $\sin\phi$, adding and dividing by z:

$$nDB - aA\sin\phi = 0 \tag{313}$$

Combining equations (308) and (313) to eliminate $\sin\phi$:

$$aB/gAn = nDB/aA$$

whence:

$$n^2 = a^2/gD \qquad n = a/\sqrt{gD} \tag{314}$$

The rate of advance of the tide and current in an ideal estuary is then $\sqrt{gD}$, the rate of advance of a frictionless progressive wave.

442. From equation (311):

$$\partial z/z = -n\cot\phi\partial x$$

The integration of which gives:

$$z = Ke^{-nx\cot\phi} \tag{315}$$

in which K is the constant of integration. When $x=0$, $K=z$. K is then the width of the estuary at the entrance, which conveniently may be designated z_0. Then:

$$z = z_0e^{-nx\cot\phi} \tag{316}$$

Or:

$$\log z = \log z_0 - (ax\cot\phi\log e)/\sqrt{gD} \tag{317}$$

If, then, the depth of an estuary is constant, and the width varies in accordance with the law expressed by equations (316) or (317), its

primary tides and currents will have a constant amplitude, and will advance up the channel at the rate of $\sqrt{gD}$.

443. To determine the amplitude of the currents, the value of $P = 1.084\, C\sqrt{rS}$ and P/S may be computed from the value of S derived from equation (301):

$$S = An = Aa/\sqrt{gD} \tag{318}$$

The value of ϕ may then be obtained from table IX, chapter V. The amplitude of the current is, from equation (308):

$$B = (Ag/\sqrt{gD}) \sin \phi = A\sqrt{g/D} \sin \phi \tag{319}$$

As shown in paragraph 338, the amplitude of current of a frictionless progressive wave, in a channel of uniform dimensions, is:

$$A\sqrt{g/D}$$

The currents in an ideal estuary are therefore less than those set up by a frictionless progressive wave:

Example.—The mean tidal range at the entrance to the estuary proper of the Delaware River, at Woodland Beach, is 5.63 feet. The mean depth of the estuary at mean tide between Woodland Beach and Philadelphia, taken from maps of about 1918, was found to be 21.5 feet. A reasonable value of the Chezy coefficient, C, is 120. Taking the tides as simple harmonic fluctuations with a speed of the M_2 component, the constants for computing the form of the ideal estuary are:

$$A = 2.815 \text{ feet}$$
$$a = 0.0001405 \text{ radians per second}$$
$$D = r = 21.5$$
$$C = 120$$

These values give:

$$S = 0.00001504$$
$$\log P/S = 5.19235$$
$$\phi = 37°21$$

The scaled width of the Delaware at Woodland Beach is 23,000 feet, the logarithm of which is 4.36173. The logarithm of the width of an ideal estuary at a point distant x feet upstream, is then, from equation (317):

$$\log z = 4.36173 - 0.000{,}0030405x$$

In figure 78, page 232, the outline of this ideal estuary is shown in broken lines on a small-scale map of the Delaware. It will be seen that the general shape of the river conforms quite closely to an ideal estuary.

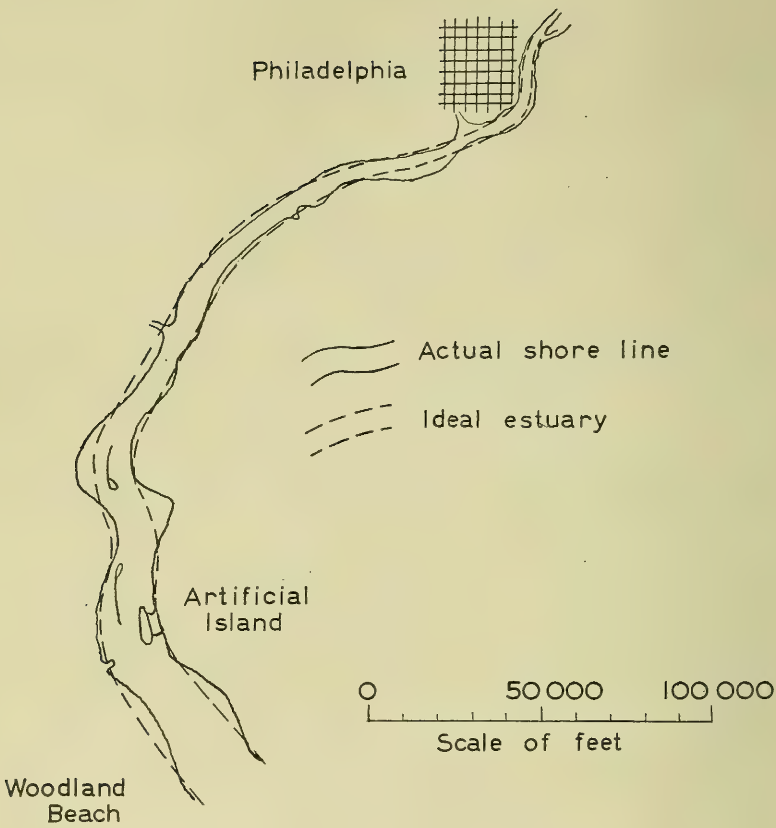

FIGURE 78.—Delaware River, Philadelphia to Woodland Beach.

444. The actual mean tidal range in the Delaware increases from 5.63 feet at Woodland Beach to 5.85 feet at the contraction at Artificial Island, then decreases to 5.09 feet at Philadelphia. The rate of advance of the tide up the ideal estuary would be $\sqrt{21.5g}=26.3$ feet per second. The actual rate of advance of the high water from Woodland Beach to Philadelphia averages 23 feet per second, and of low water, 19.5 feet per second.

The amplitude of the primary current, computed from equation (319) with the given data, is 2.09 feet per second. The actual mean current velocities at various cross sections of the Delaware, determined by cubature, have strengths generally of about 2.00 feet per second, increasing to 2.6 feet at contracted sections.

445. The equation of the tide in an ideal estuary (equation 303) shows that high water at a point distant x from the entrance occurs when $at_0-nx=0$, or when $t_0=nx/a$. Similarly equation (304) shows that the current turns from positive to negative, or from flood to ebb, when $at_1-nx-\phi=0$, or when $t_1=(nx+\phi)/a$. The interval between

high water and the turn of the current is then ϕ/a. If then the Delaware were an ideal estuary, the primary current at any station would turn from flood to ebb $37°.35/28.98=1.29$ hours after high water. The actual currents turn in this direction from an hour to an hour and a half after high water at the station.

The general characteristics of the tides and currents in the tidal portion of the Delaware conform therefore to those deduced for an ideal estuary.

446. *Prevalence of estuaries of typical form.*—The depth of a natural tidal channel is far from constant, and the variation in its width which would be required to produce currents of constant strength departs somewhat from that of the ideal estuary deduced in the preceding paragraphs. However, as a natural channel carrying a constant steady flow tends toward a general uniformity of width, a tidal estuary tends toward the funnel-shaped form of an ideal estuary. In a tidal channel which has not such a form, the currents have different strengths from section to section and the bed tends to scour where the currents are the stronger, and to fill where they are the weaker. Tidal channels in alluvial material therefore mold themselves into the typical estuary shape. The result of this process is strikingly shown in the natural channels through the tidal mud flats bordering a sheltered bay or coastal sound, in which the wave action does not cause enough littoral drift to contract the outlets. A glance at a chart of such a region, or a view from the air, shows that the many channels cut through these flats by drainage from the uplands have molded themselves into the typical estuary form, generally with a sinuous alignment.

447. Large rivers which enter the sea through an alluvial coastal plain also usually cut for themselves a typical estuary channel; unless they carry down silt and sand at a faster rate than can be molded by the tidal currents, when they maintain a generally uniform cross section to an ever-growing delta at their junction with the sea. A delta generally is found at the mouth of a silt-bearing river which, like the Mississippi, enters a sea having a small tidal range; but the burden of detritus may be sufficient to form a delta at the mouth of a river where the tidal range is large. Thus deltas are found at the mouths of the heavy silt-bearing rivers which enter Puget Sound, although the diurnal tidal range in the sound is generally 10 feet or more.

448. Many rivers enter the sea through submerged valleys, which ordinarily widen toward the sea and have the general shape of a self-made estuary. If the valley has become filled with alluvial deposits of fairly uniform consistency, the tidal flow generally has molded an estuary of typical form, often subdivided by islands and shoals. The entrance to an estuary from the open sea usually is contracted by deposits from the littoral drift along the coast line; but this contrac-

tion does not affect greatly the tides and currents inside of the entrance. In short, long natural tidal channels, other than tidal straits, are usually of estuary form; and if they are not too deep, their tides and currents ordinarily have the general characteristics of those of an ideal estuary.

449. *Effect of local contractions and enlargements upon the range and rate of the tide.*—Variations in the consistency of the bed and banks of a natural estuary result in local contractions and enlargements of the cross section, so that the strength of the current is no more uniform than is the current velocity in a natural upland stream whose bed and banks are of similar material. The consequent variation in the amplitude of the surface slope produces variations both in the tidal range and in the rate at which the tide advances up the channel. The nature of these variations may perhaps be developed most readily from a diagram.

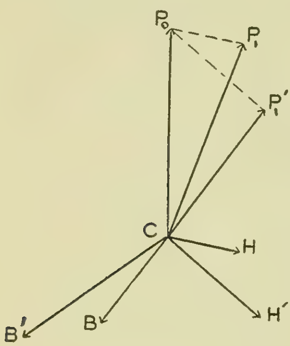

FIGURE 79.

In figure 79, CP_0 is the generating radius of the primary tide at the downstream, or initial, end of a short section of a tidal estuary. If the currents were of uniform strength, and the tides of constant range throughout the estuary, the generating radius of the tide at the upstream end of the section would be CP_1, equal in length, but lagging behind CP_0 by an angle determined by the rate of progress of the tide, $\sqrt{gH}$. The surface head in the section would then be CH, equal and parallel to P_0P_1 (par. 244); and the generating radius of the primary current would be CB, making an angle of $-\phi-90°$ with CH (par. 248). Upland inflow disregarded, this current is due wholly to the discharge at the section produced by the filling and emptying of the tidal prism upstream therefrom. The phase of the current CB has therefore a fixed relation to the phase of the tide CP_1.

If, because of a local contraction at the section, the discharge produces a current of increased amplitude CB', the amplitude of the head is increased to CH', but the angular lag, ϕ, of the current with respect to the head is decreased, so that the angle $H'CH$ is greater than $B'CB$. The generating radius of the tide at the upstream end of the section becomes CP'_1. The angle $P_1CP'_1$ is nearly or quite equal to BCB', and the angle $P_1P_0P'_1$ is equal to HCH'. It is apparent from the figure that the increase in the current results in a decrease in the tidal range and an increase in time of the tide at the upstream end, with a consequent decrease in the rate at which the tide travels through the section. The decrease in the tidal range at the upstream end tends to reduce the discharge at the section, and checks the increase in the current.

450. A local increase in the strength of the current in a section of an estuary therefore decreases the tidal range upstream, and retards the progress of the tide up the estuary. A local decrease in the strength of the current tends to increase the tidal range, and speeds up the progress of the tide. The decrease in the range as the tide passes through the contracted sections usually results in a less range in the wider and deeper sections upstream, while because of the tendency toward an increase in the wide and deep sections, the range of the tide normally increases as it approaches a contracted section. The larger tidal ranges are therefore found ordinarily at the contractions, and the smaller ranges in the wide and deep sections of an estuary. The advance of the tide up the estuary is more rapid where the tidal range is increasing than it is when the range is decreasing.

451. *Deep channels of estuary form.*—Submerged valleys, unfilled by alluvial deposits, afford some long closed tidal channels, flaring toward the entrance like an estuary, but so deep that the frictional resistance to flow is very small. The convergence of the shores of the ideal estual becomes less as the depth increases and the frictional resistance to flow decreases. In a channel so deep that the flow is essentially frictionless, ϕ is so close to 90° that cot ϕ is practically zero. The width of the ideal estuary then closely approximates, from equation (316):

$$z = z_0$$

The ideal estuary becomes an endless channel of uniform width and depth. It follows, therefore, that in a closed channel of finite length the tides maintain a constant range, and the currents a constant strength, only when the channel is so shallow that the frictional resistance to flow is material. When its depth is so great that the frictional resistance is negligible, the tides and currents take the general characteristics of those produced by frictionless flow in a closed canal, discussed in chapter VII. The wave lengths of the principal tidal components become so long in a deep channel (par. 326) that the length of nearly all natural channels is but a fraction of these wave lengths, and the range of the tide characteristically increases from the entrance to the head. If the effective length of the channel approximates one-quarter of the component wave lengths corresponding to its depth, the range of the tide at the head of the channel may be very large. In timing, the tides approach the condition of a stationary wave, which rises and falls simultaneously.

452. *Tides and currents in the Bay of Fundy.*—The Bay of Fundy, on the Atlantic coast of Canada, just north of the State of Maine, affords the outstanding example of the heights to which the tide may rise at the head of a fairly deep natural channel. An interesting

description of the tides in the bay is given by Marmer in "The Tide", from which the figures herein have been abstracted. The bay extends 170 miles inland and there subdivides into two comparatively small and shallow branches. The entrance to the bay is 85 miles wide, and has a mean depth of 280 feet. The bay gradually narrows to 30 miles at the junction of the branches, where the mean depth becomes 130 feet. The mean tidal range increases from about 13 feet at the entrance to 40 feet or more at the heads of the two branches, reaching 50 feet at spring tides. This is the greatest tidal range in the world. The midchannel currents at the entrance to the bay have a strength of 1½ knots. The tide tables indicate that high water progresses 90 miles in 15 minutes in the deep water in the main part of the bay, but the progress of the tide slackens in the shallower branch channels. The current turns nearly at high and low water.

The mean depth in the bay may be taken at 240 feet. The corresponding wave length of the principal lunar semidiurnal component, M_2, is 663 miles (par. 326). The length of the bay is therefore nearly one-quarter of this wave length. As shown in paragraph 346, this is a critical length of a closed canal of uniform dimensions, at which the tides are limited only by frictional resistance. While the analogy is far from accurate, it affords an explanation for the great tidal range at the head of the Bay of Fundy.

453. *Other examples of the increase in the tidal range in deep channels.*—The Gulf of California, inside of the peninsula of Lower California, is a deep channel extending inland over 700 miles from the Pacific Ocean to the delta cone at the mouth of the Colorado. The mean tidal range decreases from 4 feet at the entrance to 3 feet in a zone about 300 miles up the gulf, and then increases to 22 feet at the mouth of the Colorado. In Cook Inlet, in Southwestern Alaska, a deep, funnel-shaped channel about 200 miles in length, the mean tidal range increases from about 12 feet at the entrance to 30 feet at the head. Long Island Sound affords another and often quoted example of the increase in the tidal range in a fairly deep closed channel whose length and depth have the relation which should lead to this increase. The sound has a prevailing depth of 65 feet and a length of 70 miles. This length is approximately one-quarter of the wave length of the principal semidiurnal tidal components at the given depth. The entrance from the sea, at the eastern end, is contracted by a chain of islands, in the passage between which the currents are strong, but these passages are so deep and so short that the currents do not appear to produce any considerable head. Inside the entrance the tidal currents are weak. As is to be expected under these conditions, the tidal range increases from 2½ feet at the eastern entrance to 7½ feet at the western end of the sound. High water travels through the sound in about half an hour.

In contrast, the Lynn Canal, in southeastern Alaska, is a narrow fiord, 80 miles long, with a prevailing depth of 1,000 feet or more. Its length is but one-twentieth of the wave lengths of the principal semidiurnal components at the depth. The mean tidal range increases from 12.6 feet at Barlow Cove, near the entrance, to but 14.6 feet at Skagway, at the head of the fiord. The tide is so nearly a stationary wave that high water at the head occurs but 5 minutes after high water at the entrance.

454. *Effect of fresh water discharge.*—The fresh-water discharge of a river increases the ebb currents in its tidal reaches and decreases the flood currents. In the wide part of a typical estuary, nearer its junction with the sea, the tidal discharge from the storage in nearly the whole of its tidal prism may be so much greater than the fresh water discharge that the latter has but little effect upon the currents

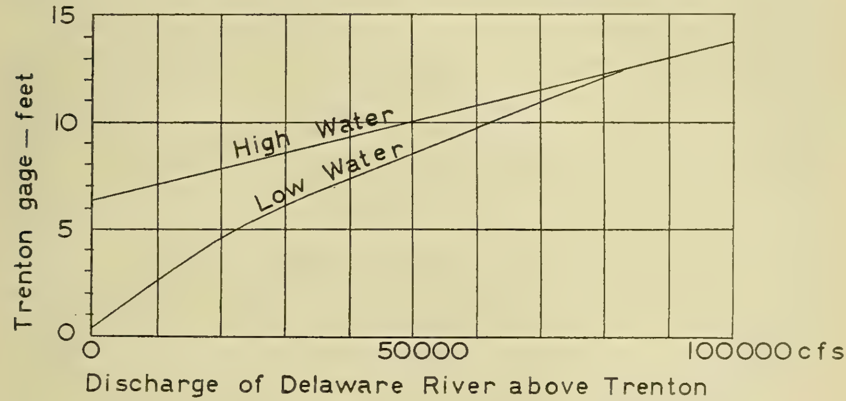

FIGURE 80.—Relation of high and low water to fluvialdischarge at Trenton, N. J.

The discharge from tidal storage steadily diminishes upstream, the ebb currents increase, and the flood currents decrease until a point is reached at which the flood current disappears. Above this point, the current fluctuates in velocity, but does not change direction. The mean tide elevation in the river slopes upward from the sea at an increasing rate as the ebb currents become the stronger. If the river has so ample a cross section that the slope is small, the fluctuations of the tide may extend far up the stream, diminishing as the backwater from a dam diminishes, until at some point the tides disappear, and with them the tidal storage, and the last traces of tidal fluctuation in the river current.

Quite obviously, the range of tides in the upper reaches of a tidal estuary diminishes when the fresh-water discharge increases, and may disappear when the river is in flood; as the backwater from a dam diminishes and eventually disappears with the increasing river discharge. The observed heights of high water and of low water at a tidal station may be plotted against the upland discharge to afford a

diagram showing the effect of the discharge upon the tides. Such a diagram for the Delaware River at Trenton, near the head of tide, prepared from selected monthly mean high and low waters and discharges during the period 1922 to 1926, is shown in figure 80.

455. The tidal part of many of the larger streams entering the Atlantic Ocean in the United States, terminates abruptly in the rapids at which these rivers drop into the Coastal Plain, or into the submerged valleys in which their tidal courses lie. The upstream tidal reaches usually have the capacity to carry the ordinary river discharge. During periods of low discharge the flow in these reaches becomes almost entirely tidal, and in many cases the tidal range then increases toward the head of tide, instead of gradually decreasing upstream.

456. *Distribution of the currents due to fresh-water discharge.*—As fresh water has a less specific gravity than salt water, the salt water usually underruns the fresh at the turn of the current, so that the ebb continues on the surface while the flood current is running in beneath. Numerous meter measurements made at various depths at the mouth of the Hudson River show that the strengths of the ebb currents generally are relatively less than the strengths of the flood in the deeper part of the channel (Special Publication No. 111, U. S. Coast and Geodetic Survey).

457. *Difference in tidal range on the opposite sides of a wide estuary because of the earth's rotation.*—Unexpected as it may seem, the rotation of the earth produces a measurable difference in the tidal ranges on the opposite sides of a wide estuary. Consideration will show that the earth rotates under the moving water in the channel, as it rotates under a Foucault pendulum. At a place whose latitude is λ, the rate of rotation is 360° sin λ per (siderial) day or 0.000,072,9 sin λ radians per second. In the northern hemisphere the currents, if unrestrained, would rotate clockwise at this rate with respect to the earth. Since the direction of the current in a channel is restrained by the banks, the rotation sets up a slight transverse slope of the water surface.

Designating the rate of rotation of the earth about its axis, in radians per second as ω (omega) and the velocity of the current in the direction of the channel by v, the transverse component of the velocity, due to the earth's rotation, if unrestrained, would be $\omega v \sin \lambda$. Since the steadily exerted force required to restrain a body from motion at a given velocity is twice that necessary to accelerate it to the velocity, the pressure acting on each unit of mass of the flowing water, to restrain it in the direction of the channel is $2\omega v \sin \lambda$, and the transverse slope to produce this pressure is $2\omega v \sin \lambda/g$. The difference in level between the two banks of the channel is then $2\omega vz \sin \lambda/g$, z being the width of the channel.

458. Ordinarily the flood current in an estuary is near its strength at high water, and the ebb current near its strength at low water.

Looking *upstream*, as is customary in regarding channels which lead in from the sea, the rotation of the earth therefore tilts the water surface upward to the right at high water, and upward to the left at low water; with the consequence that the tidal range on the right (ascending) bank is greater than that on the left bank by $4\omega vz \sin \lambda/g$. Since ω=0.0000729, and the value of g is not far from 32.16, this increase in range becomes, when z is expressed in statute miles and v in feet per second, 0.05 $vz \sin \lambda$. If z is expressed in nautical miles of 6,080.2 feet, and v in knots, the difference in range is $2.92vz \sin \lambda/g$, or 0.09 $vz \sin \lambda$.

459. The observed differences in the tidal ranges on the two banks of a wide estuary conform fairly well with this formula. Thus at the entrance to Delaware Bay the distance between the two shores is 10 nautical miles, and the average current at high and low water is about 1 knot. The entrance is at latitude 38°20, whose sine is 0.62. The difference in range between the two shores from the formula is 0.56 feet while the observed difference is 0.6 foot. At the head of the bay the width is 4 nautical miles, the current is 1.3 knots, and the latitude 39°23, giving a calculated difference of 0.3 foot, while the actual range on the right, ascending, bank is 0.2 foot greater than on the left. A similar concordance with the formula is observed in other tidal waters.

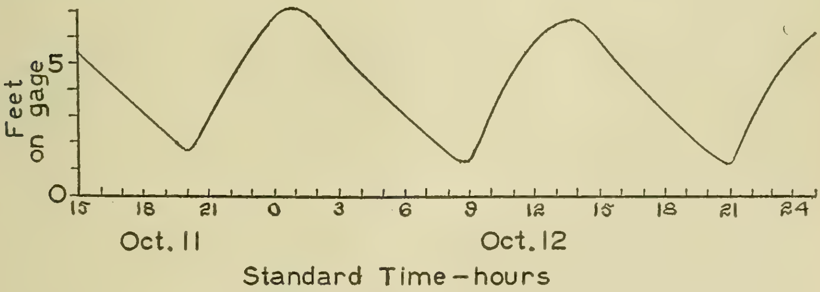

FIGURE 81.—Tide curve of Delaware River at Philadelphia October 11–12, 1924.

460. *Overtides and overcurrents in an estuary.*—As is to be expected, the tide advances more rapidly up an estuary, or any long closed tidal channel, at high water, when the depth in the channel is the greatest, than it does at low water when the depth is the least. The further a tidal station is up an estuary, the earlier is the time of high water with respect to the time of low water. The time interval from low water to high water, or the "duration of the rise," steadily becomes less as the distance of the station from the entrance increases, and the time interval from high water to low water, or the "duration of the fall" becomes greater. The tide curve takes the typical saw-tooth form exemplified by the tides on the Delaware at Philadelphia, shown in figure 81.

As pointed out in paragraph 155, these deformations of the tide curves are reproduced by overtides and compound tides whose periods are multiples or sums or differences of the periods of the principal tidal components. The corresponding deformation of the velocity curve at a tidal station is the accumulated effect of the tidal distortions at the stations upstream upon the rate of tidal storage and release of water, and is consequently greater than the deformation of the tide at the station. A typical shape of the velocity curve at Philadelphia was shown on figure 50, page 155.

461. It should be noted that the deformation of the tides and currents as they travel up an estuary is due primarily to the difference between the depth in the channel at high and at low tide, and depends therefore on the ratio of the tidal range to the mean depth. Although the deformation may be increased because of the stronger ebb and weaker flood currents resulting from fresh-water discharge down the estuary, the latter is not the essential reason for these deformations.

462. *Slope of mean river level.*—Since in each section of an estuary the ebb current runs out at the lower tidal stages and the flood current runs in at the higher stages, the frictional resistance to the flow of the ebb is greater than that to the flow of the flood current. As a consequence the mean river level in an estuary has an upward slope from the sea, even though the fresh-water flow is negligible. In a channel deep enough to be navigable by ocean shipping at low tide, this slope is very small. In a shallow estuary it may be considerable.

463. *The tidal bore.*—The successive instantaneous profiles in a tidal estuary show the water surface advancing up the channel as a long wave, outwardly resembling, in a general way, the advance of a wind wave toward the shore. In nearly all estuaries the slope of the front of the advancing wave is very small. This slope steepens as the depth of the channel decreases, and as the currents increase with the rate of rise of the tide. The rate of rise of the tide rarely is sufficient to create an excessive slope on the front of the wave even when the estuary is so shallow that much of its bed runs bare at low tide; but if the range of the tide is so large that its rise is exceptionally rapid, and if the fast rising tide encounters a strong outflowing current, the advancing wave may trip and break, like a wind wave breaking on the shore. The incoming tide then rushes up the shallows in a breaking wave, generally called the *tidal bore*, but otherwise known as the "Aegre" or "Hygre" in England, the "Mascaret" in France, and the "Proroca" in South America. A bore is formed in one of the shallow tidal branches at the head of the Bay of Fundy; it forms also in the mouth of the Colorado, at the head of the Gulf of California; and in the shallow waters at the head of Cook Inlet, Alaska; but does not appear to form in any other estuaries on the North American Continent. Because of the large tidal range at many localities on the

coasts of England and France, bores occur in a number of the shallower estuaries of these countries. The most noteworthy tidal bore reported is that in the Tsien-Tang-Kiang River, which enters Hangchau Bay, some distance south of Shanghai, in China. The tidal range in the mouth of the river, at ordinary spring tides, is given as 25 feet, sometimes reaching as much as 34 feet. The bed of the river runs nearly bare at low tide. The incoming tide advances in a breaking wave which is described as from 8 to 12 feet in height, rushing with a loud noise up the estuary at a rate of 14 miles per hour (Wheeler, Practical Manual of Tides and Waves, pp. 142–144).

EFFECT OF ARTIFICIAL CHANGES IN AN ESTUARY

464. *Comprehensive enlargement.*—A comprehensive enlargement of the channel in a long tidal estuary, to afford greater depth and width for navigation, generally increases the tidal range in the upstream reaches and increases the rate at which high and low water travel up the channel. The increase in range depresses the plane of mean low water, and other low water datums. Additional excavation is therefore required to afford the projected increase in depth at a designated low water datum. Thus, after the navigation channel in the upper part of the estuary of Hudson River was deepened from 14 feet to 27 feet at mean low water, the mean low water datum at the head of the improvement was lowered by a foot. The increase in the depth of the navigation channel between Philadelphia and Trenton, from 12 feet to 25 feet, also depressed the low water datum at Trenton by a foot.

465. *Contractions.*—A radical local contraction of an estuary by training works, piers, or land reclamation, decreases the tidal range upstream. The consequent reduction in the volume of the tidal prism decreases the currents below the contraction and tends to increase the tidal range at and below it. The removal of a marked local contraction at midlength of a long estuary similarly increases the storage and release of water upstream, increases the currents below the contraction and may decrease the tidal range at and below the site. A decrease of about half a foot in the mean tidal range on the Delaware at and below Philadelphia, shown by a comparison of the tide gage records prior to 1890 with those after 1900, usually is ascribed to the contractions at the extensive training works which were constructed in the lower part of the estuary during the interval. The decrease in range may have been due partly, as well, to the major enlargement of the river at Philadelphia during this period. This enlargement included the removal of several islands which had so contracted the cross section as to create excessive currents. The enlargement was followed by a considerable increase in the tidal range at the head of the estuary.

466. *Dams.*—The construction of a dam across an estuary, to maintain the upstream reaches at low tide level in the interest of the reclamation of tidelands, or at high tide level in the interest of navigation, recreation, and sightliness, obliterates the tidal storage upstream. The tidal currents in the downstream reaches are diminished, and disappear at the dam. The tidal range at the dam is increased by an amount dependent upon the length and depth of the remaining part of the estuary. The accumulation of silt in the channel below the dam ordinarily is to be expected.

467. *Character of computations of the effect of enlargements or contractions.*—In the preceding chapter a method was developed for computing the tides and currents in an artificial channel of such regular dimensions that the Chezy coefficient in the successive subsections could be selected with sufficient assurance from precedent. A somewhat different problem arises in estimating the changes in the tidal ranges and currents that may be expected from projected enlargements or contractions which merely will modify, without essentially changing, the characteristics of the flow in a long tidal channel. The latter problem is somewhat analogous to an estimation of the changes in the slopes and currents of an upland river because of similar enlargements or contractions. In both cases the immediate effect upon the currents at the locality is easily determined, but a reliable computation of the consequent effect in other parts of the channel can be secured only from an elaborate and painstaking analysis of the existing flow in the successive subsections, based on adequate survey and records.

468. Fortunately, a computation of the changes in the tidal ranges, tidal datums, and currents because of projected enlargements or contractions of a tidal channel is called for but rarely. If a closed channel is relatively short, the datum throughout it can be taken as the established datum at the entrance, whatever the scope of the proposed improvement; for while, as shown in paragraph 427, the tidal range at the head of such a channel may be greater than at the entrance, the increase in range and the lowering of the low water datum at the head of the channel generally is too small to be of real consequence.

Projected local enlargements or contractions of a long tidal estuary rarely are so extensive that any material change in the low water datums need be apprehended. Because of the daily variation in low water, a change of even a foot in the low water datum, resulting from major channel enlargements, is not immediately apparent. An early redredging of the channel often is required in any event to remove material which has slid in from underwater slopes or has been deposited from other sources. If a projected improvement has been cut so close that a shortage of a foot or so in the depth at low water is of any real consequence to shipping, its further enlargement is to be foreseen.

Except, perhaps, for providing initially a reasonable margin of increased swept depth over areas that must be drilled and blasted, the sensible procedure in nearly every case is to lay out the work from existing low water datums, and to determine any required changes in these datums by direct observation after the improvement has been made. Only in most exceptional cases will doubt or controversy over the consequences of the effect of enlargements or contractions justify a prior computation. In the following paragraphs, an outline is suggested of computations which should afford results in which some confidence may be placed when the flow is essentially tidal. If the fresh-water flow dominates the currents and tides, recourse to a hydraulic laboratory might be necessary.

469. *Computation of changes in mean low water datum.*—If the tides are of the semidiurnal type and if the ordinary fresh-water flow is small in comparison with the tidal flow, or if the adopted mean low water datum is established from the tides during periods in which the fresh-water flow is inconsiderable, the changes in mean tidal range at stations along an estuary, resulting from proposed enlargements or contractions of the channel, should be substantially proportional to the changes in the primary tides corresponding to the mean tidal fluctuations. The mean primary tides before improvement may be determined from the tide records, and those after improvement computed by the formulas developed in chapter VIII, paragraph 422, with coefficients derived from the corresponding primary currents before improvement.

470. *Primary tides and heads before improvement.*—To afford the requisite data for the computations, tide gages must be established at suitable stations from the head of tide to a point at which the cross section of the estuary is so large that the effect of the improvement upon the currents will become too small to be considered. These stations should be placed at the more marked changes in the cross section of the estuary, and at such distances from each other that the water surface between them will not depart materially from a plane surface. They establish the ends of the subsections into which the channel is to be divided. From the tide records during a period of 15, or preferably 29, consecutive days of low upland flow, average tide curves are prepared for each station as described in paragraph 304, and the corresponding primary tides computed from the heights at successive lunar hours as explained in paragraph 360. The coordinate amplitudes of the primary tides are then determined. Their differences between the successive stations give the coordinate amplitudes of the primary heads between the stations, from which the amplitudes, H, and initial phases, $H°$, of the heads in the subsections are determined.

471. *Primary currents before improvement.*—The velocity stations are midway between the established tide stations, and the storage stations, generally, should be midway between the velocity stations. The coordinate amplitudes, $A \sin \alpha$ and $A \cos \alpha$, of the primary tide at the storage stations are obtained by linear interpolation between those at the established tide stations. Taking any representative cross-section area of the estuary as a base, M_0, the values of $I = aU/M_0$ between the velocity stations are computed from the surface areas U at mean tide, a being 0.0001405 radians per second. The summation from the head of tide of the values of $IA \sin \alpha$ and $IA \cos \alpha$ then gives the values of $(B/m) \sin \beta$ and $(B/m) \cos \beta$ at the velocity stations, from which the values of β and B/m may be determined. The average or effective cross-section area, M, in each subsection may be determined from a consideration of a sufficient number of plotted actual cross sections. The multiplication of B/m by $m = M_0/M$ then gives the amplitude B of the primary current in the subsection before improvement.

472. *Subsection coefficients before improvement.*—Since the values of H^c and β in each subsection have been found, the angular lag, ϕ, of the current is determined from the relation expressed in equation (290), paragraph 373:

$$\phi = H^\circ - \beta - 90^\circ.$$

This value should also satisfy the relation, from equation (289):

$$\sin \phi = Bla/gH.$$

While the values of ϕ computed from these two equations should not be widely apart, a complete agreement cannot be expected. The value of ϕ should therefore be computed from equation (290), the length, l, in equation (289) taken as the virtual length of the subsection, and the value of the coefficient la/g computed from the relation:

$$la/g = (H/B) \sin \phi.$$

The value of the coefficient p in each subsection is determined from the relation, from equation (288):

$$p = B \tan \phi.$$

473. *Subsection coefficients after improvement.*—If the area, U, of the water surface between any of the velocity stations is changed by the proposed improvement, the coefficient $I = aU/M_0$ must be recomputed for the value of M_0 originally chosen. The effective cross section $M,'$ in each subsection after improvement, determined by a procedure paralleling that used in the selection of the value, M, before improve-

ment, gives the value of $m'=M_0/M'$. The coefficient la/g remains unchanged. Since $p=(3\pi/8)(a/g)C^2r$ its value after improvement is

$$p_1=p(C^2r/C_1^2r_1)$$

in which C and C_1, r and r_1 are the values of the Chezy coefficient and the hydraulic radius before and after improvement. Often C_1 may be taken as the same as C, so that

$$p_1=pr/r_1.$$

The value of r_1 should be computed by a procedure paralleling that used in the computation of r.

474. *Completion of computation.*—The primary currents that would be produced in the improved channel, if the tides were unchanged, are first computed. If the values of I are not changed, the phases, β, are those already determined, and the amplitudes, B, are derived by multiplying the values of B/m, previously found, by the new ratio m'; otherwise the values of $IA \sin \alpha$ and $IA \cos \alpha$ are recomputed from the primary tides at the storage stations and the values of (B/m') $\sin \beta$ and $(B/m') \cos \beta$ found by their summation from the head of tide. The coordinate amplitudes of the heads in the subsections corresponding to these currents are then computed from the values ascertained for la/g and p_1; the corrected coordinate amplitudes of the tides derived therefrom; and the process repeated until the tides and currents are in satisfactory concordance. The elevation of mean low water after improvement is then found by multiplying the mean semirange of the tide at the station, as established by comparison or otherwise, by the ratio of the computed amplitudes of the primary tides after and before improvement, and subtracting the result from established half-tide level.

475. *Computation of changes in mean lower low water datum.*—If the tides are of the mixed type, and the adopted datum is mean lower low water, the changes resulting from an extensive channel improvement might be computed on the assumption that the ratio of the mean range to the diurnal range will remain the same after and before the improvement. The primary tides after and before improvement could then be computed as outlined in the preceding paragraphs. The ratio of their amplitudes at a station would then give the ratio of the elevations of mean lower low water below established half-tide level after and before the improvement.

476. *Application to the approach to a sea-level canal.*—The changes that may be expected in the tides and currents in a confined approach to a sea-level canal because of the flow in and out of the canal entrance, could best be computed by determining the subsection coefficients in the approach channel from tidal observations made before the canal

was opened. The computations of the tides and currents in the canal would then be extended through the approach channel. The coordinate amplitudes of the tides at the storage stations in the approach channel used in the initial computation would be determined from the primary tides at these stations derived from the observations.

477. *Computation of the effect of dams or other works decreasing the tidal prism.*—The effect of a projected dam, or of other works which would decrease materially the area of the tidal prism, upon the currents in an estuary or other channel, is definitely ascertained by making a cubature of the channel with the prism unimpaired, and a cubature from the same tides with the reduced prism. While some increase in the tidal ranges below the dam is to be anticipated, the counterbalancing effect of the increase ordinarily is not sufficient to warrant consideration. Similarly any question that might arise on the effect of the excavation of a considerable tidal basin in the upper reaches of an estuary may be settled by comparative cubatures.

TIDAL INLETS

478. *Prevalence of inlets.*—The littoral drift of sand and shingle along the seacoast tends to build up beaches across the entrances to the identations of the shore line. This process has formed the coastal sounds and lagoons which are the prevailing feature of the coast line of the United States from Maine to the Rio Grande, and which are found occasionally on the Pacific and even the Alaskan coasts as well. Inlets into most of these sounds are preserved by the currents set up in these channels by the filling and emptying of the tidal prism. The entrances to nearly all tidal estuaries are similarly contracted by littoral drift, sometimes sufficiently to produce typical inlet channels.

479. *Typical shape of inlet channels.*—The material carried by littoral drift into an inlet channel is removed by the currents through the inlet. At an inlet into a coastal sound, it is deposited in fan-shaped bars in the approaches both from the sea and from the sound. A typical natural inlet channel has a deep, narrow gorge through the barrier beach, from which it spreads in both directions with diminishing depth. The sea approach to the gorge often is through ill-defined and shifting channels between sand bars. In the sheltered waters of the sound, the bars may even build up into islands. If the basin is small and shallow the approaches may become so prolonged and constricted that the currents are no longer sufficient to cope with the encroaching littoral drift, and the inlet closes. The entrance to a large estuary, in which the ebb currents predominate, is often encircled by a crescent-shaped bar, well out to sea.

480. *Hydraulics of inlet channels.*—The improvement of tidal inlets, to afford stable and adequate channels for navigation across their ocean bars, or for other purposes, has an important place in harbor

engineering. It often is accomplished by constructing jetties to impound the littoral drift and to concentrate and direct the currents over the sea bar. In the design of these works consideration must be given to the cross section of the channel that can be maintained by the tidal flow, and to the effect of a contraction or enlargement of the channel on the tidal ranges in the basin and the consequent currents through the inlet. A mathematical analysis of the relation between the capacity of an inlet channel, the tides in a basin of given surface area, and the currents in the inlet, is necessarily based on the assumption that the inlet channel is of determinable length and regular cross section, and the basin so deep and of such limited area that its tides have the same timing and the same amplitude at all points. The approach channels of a natural inlet depart so far from these ideal conditions that the computation of the tidal currents in them is as uncertain as is the computation of the currents in an irregular shoal reach of an upland stream. Even a channel between parallel jetties is apt to have an unpredictably irregular cross section. Furthermore, the tides at stations on a wide and comparatively shallow basin do not rise and fall simultaneously, but become progressively later the more distant the station from the entrance. Space will not therefore be taken for a mathematical analysis of inlet tides. Their outstanding characteristics may be inferred from elementary hydraulic relations.

481. It is fairly evident that the frictional resistance in the constricted channels through an inlet must reduce the amplitude of the tides in the basin, and delay the rise and fall of these tides, so that high and low water in the basin are later than in the sea off the inlet. If the constricted channels of the inlet are relatively short, the currents must become excessive before the friction head can be sufficient to have any material effect upon the tides in the basin. Stable short inlets through erodible material therefore are usually so large that the tidal range inside the inlet is practically the same as that outside. If the improvement of such an inlet is so designed that the discharge, determined from a cubature of the recorded tides in the basin, will not produce excessive currents, no apprehension need be felt that the improvement will have any material effect upon the tides in the basin, or reduce the tidal discharge. Again, the straightening and deepening of inlet approach channels which have become so filled and prolonged as to throttle the tidal range in the basin may be expected to increase the currents in these channels, and increase the tides in the basin, until the channels have been given a sufficient capacity to nearly equalize the tidal range in the basin and in the sea. In either case the maximum cross section of a self-maintaining channel is determined by the volume of the unimpaired tidal prism in the basin.

482. *Observed relations between the volume of the tidal prism and the capacity of inlets on the Pacific coast of the United States.*—A compilation, made by Prof. M. P. O'Brien of the University of California, printed in Civil Engineering, May 1931, shows that the area, at mean tide, of the cross section at the throat of the entrances to the estuaries and bays on the Pacific coast of the United States, conforms quite closely to the relation:

$$M=1{,}000V^{0.85}$$

in which M is the area of the entrance in square feet, and V is the volume of the tidal prism of the basin between MLLW and MHHW, in square mile-feet.

It should be observed that the tides on this coast are of the mixed type, whose sequence is such that lower low follows higher high water. The diurnal range, from MLLW to MHHW, therefore affords a measure of the stronger ebb currents.

483. A study made in office of the Pacific Division, United States Engineer Department, by Mr. Grimm, principal engineer, shows that the area of the cross section over the *ocean bars* of the larger estuaries of the Pacific coast of the United States, at MLLW, is from 1.04 to 1.26 square feet per acre-foot of tidal prism in the basin between MLLW and MHHW. The corresponding average strength of the ebb currents is about 2 feet a second.

484. *Overcurrents in inlets.*—The currents in some inlets are much distorted by the overcurrents produced by the variation in the area of the water surface in the basin, and in the area of the cross section of the inlet, with the rise and fall of the tide. The curve of the flood velocities in such an inlet may rise rapidly to a maximum, fall off, and again rise to a second maximum, before turning to the ebb. The ebb currents may go through a similar variation.

CHAPTER X

OFFSHORE TIDAL CURRENTS, REDUCTION OF CURRENT OBSERVATIONS, AND CURRENT PREDICTION

OFFSHORE TIDAL CURRENTS

485. *Rotary tidal currents.*—The tidal flow heretofore considered has been that in a confined channel, in which the currents periodically reverse their direction and pass through zero at each reversal. A consideration of the tide producing forces, developed in chapter I, shows that their direction is rotary rather than reciprocating. As is perhaps to be expected, the action of these forces on the whole mass of water in the oceans tends to produce rotary movements of the current at offshore tidal stations. At such stations, the currents usually veer around the compass during the tidal cycle, and have no periods of slack water. These are called *rotary currents.* At most offshore stations in the Northern Hemisphere the direction of the current turns clockwise, and in the Southern Hemisphere, counterclockwise. The velocity usually varies during the semidiurnal tidal cycle between two maxima, in approximately opposite directions, and two minima whose directions are nearly at right angles to the directions of the maximum velocities.

486. *Nontidal currents.*—The periodic tidal currents at offshore stations are generally weak and may be much modified by *permanent* currents of fairly constant strength and direction produced by the circulation of ocean waters, and by *temporary* currents due to winds and other meteorological causes. The Gulf Stream and the Japan Current are well known permanent currents.

487. *Polar current diagrams.*—Offshore currents are conveniently represented by laying off the current strengths at say hourly intervals on radiating lines (radii vectores) drawn from a common center (pole) in the direction of the current. The curve through the ends of these vectors is the polar curve of the current. The time is marked on the

vectors. Since the directions and velocities of the current are repeated, with some variation, at intervals of the periods of the tidal cycles, and since high and low water at any tidal station in the same region are repeated at nearly the same intervals, the times marked on the diagram generally are referred to the times of high and low waters, or of the principal current phases, at a well-established tidal station.

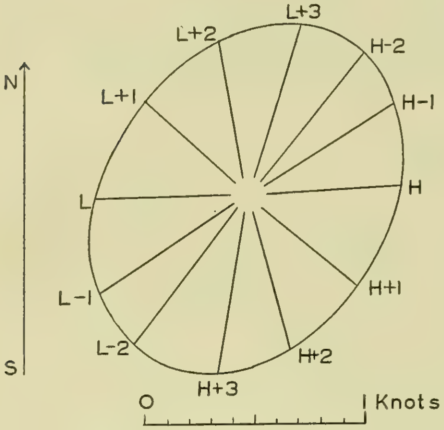

FIGURE 82.—Mean current curve for Nantucket Shoals Lightship, referred to tides at Boston.

488. *Shapes of polar current curves.*—In regions where the tides are of the semidiurnal type the currents are nearly identically repeated during each successive semidiurnal tidal cycle, and the current curve usually has an elliptical shape, exemplified by the mean current curve at Nantucket Shoals Lightship, figure 82, taken from the Manual of Current Observations, United States Coast and Geodetic Survey (Special Publication No. 215). The times marked on the diagram are referred to the times of high and low water at Boston. Thus "H−2" marks the current 2 mean solar hours before high water at Boston, and "L+3" the current 3 hours after low water at Boston.

489. In regions where the diurnal inequality of the tides is considerable, the currents during the two semidiurnal cycles have a corresponding inequality, and the daily tide curve describes a double loop, exemplified by the mean current curve at Swiftsure Bank Light-

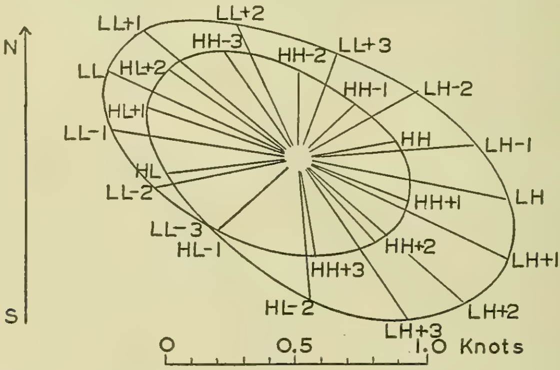

FIGURE 83.—Tidal Current Curve, Swiftsure Bank Lightship. Refer-red to predicted time of tide at Astoria, Oreg.

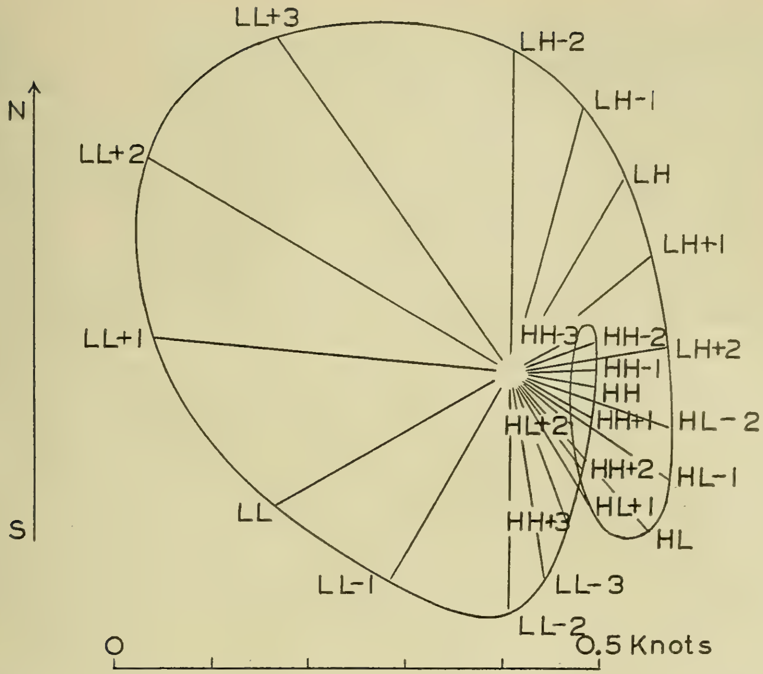

FIGURE 84.—Tidal Current Curve, San Francisco Lightship. Referred to predicted time of tide at San Francisco (Golden Gate), Calif.

ship, off the entrance to the Strait of Juan de Fuca, figure 83, taken from the same source.

The times of the currents on this diagram are referred to higher high water (HH), lower low water (LL), lower high water (LH) and higher low water (HL) at Astoria, Oreg.

490. At some tidal stations the second current loop may become very small, as shown in the mean current curve at the San Francisco Lightship, 10 miles off the entrance to San Francisco Bay (fig. 84). The current swings through a nearly complete circle, and then swings backward and forward through a limited arc before it resumes its swing around the compass. The behavior of the currents at this and other similar stations varies greatly with the declination of the moon. At the time of equatorial tides the curve has two nearly equal loops and the current swings around the compass twice during the day. At the time of tropic tides, the secondary loop becomes very small or vanishes altogether, and the current makes but one daily swing entirely around the compass.

491. *Combination of constant and rotary currents.*—In figure 85, PP_1P_2 is the current curve, and O, at its geometrical center, is the pole, of a rotary tidal current. The vector OP then represents the direction and velocity of the tidal current at a given time.

If a constant (nontidal) current at the station has the direction and strength $O'O$, the resultant of the tidal and constant currents at the given time is $O'P$. Since P may be any point on the tidal curve, the current curve of the resultant is the same as the curve of the rotary tidal current; but the pole is shifted to O'. This shift is in the direction opposite to that of the constant current and through the distance representing its velocity. If the velocity of the constant current exceeds the tidal, when the latter has an opposing direction, the pole O shifts to a point O'', outside of the curve. The direction of the resultant current then swings to and fro in the limited arc between the tangent vectors $O''P_1$ and $O''P_2$.

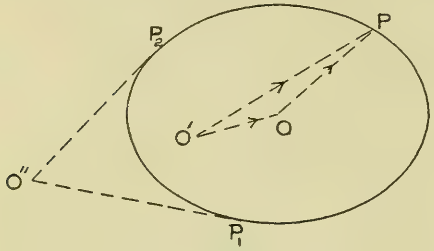

FIGURE 85.—Combination of constant and rotary currents.

The position of the pole of the diagram in figure 84 shows a pronounced constant set of the current toward the north west at the station.

HARMONIC ANALYSIS AND PREDICTION OF TIDAL CURRENTS

492. *Current tables.*—Advance information of the time at which the currents in tidal waterways will change direction, and will reach their strength at the flood and ebb; and of the maximum velocities of the surface currents in the navigation channel at each flood and ebb, is of such value to navigators that yearly current tables giving this information for the tidal waterways in and adjacent to the United States are prepared and printed by the United States Coast and Geodetic Survey. The tables give the predicted times of slack water as the current turns from ebb to flood, or "slack before flood," and from flood to ebb, or "slack before ebb," and the times and velocities of the maximum flood and ebb currents, on each day of the year, at a considerable number of reference stations. They also give the corrections to be added to or subtracted from these times to obtain the predicted times at a large number of secondary stations, and the factors for reducing the predicted current strengths at a reference station to those at the secondary stations. Most of the stations listed are in confined channels at which the currents are of the reversing type. Needless to say, the velocities in the tables are not the mean velocities in the cross section of the waterway, which have heretofore been dealt with, but are the surface velocities at definitely located points or stations, so selected as to represent the currents which will be encountered in the navigation of the fairway.

493. *Preparation of current tables.*—The fluctuations of the tidal currents, like the tides, are caused by the tide producing forces of the

moon and sun. The currents at any station may therefore be resolved into harmonic components of constant amplitudes, whose speeds are the same as the speeds of the tidal components. The mean amplitudes and the epochs of the several current components at the selected reference stations are determined from an harmonic analysis of the actual current velocities and directions at the station, measured by float or current meter, at hourly or half-hourly intervals for a sufficient number of days. The dials of a tide-predicting machine are set at the component current amplitudes reduced to the current year, and at the component phases at the beginning of the year; and the current predictions at the reference stations are run off like the predictions of the tides. At stations where the tide is of the rotary type, the harmonic constants of the east-west and north-south components of the tide may be similarly computed, their resultants in the prevailing direction of the maximum and minimum currents ascertained, and the predicted times and strengths of the currents in these directions run-off from the machine.

494. The corrections to be applied to the predicted times and strengths of the current at a designated reference station to obtain those at a secondary station are derived from the average intervals between a lunar transit and the times of slacks and strengths at the two stations, and the average tidal current velocities at the strengths of the current. The compilation of this data is termed the nonharmonic reduction of the observations, as distinguished from the harmonic reduction by which the harmonic constants at the reference stations are obtained.

495. *Accuracy of tidal current predictions.*—The actual times of slack or strength of the current at a station occasionally differ by as much as half an hour from the predicted times, and in rare instances by as much as an hour. Comparisons of the predicted and observed times show that more than 90 percent of the slack waters have been within half an hour of the predicted. Both the times and the strengths of the currents in tidal estuaries may be greatly altered by unpredictable variations in the fresh-water discharge, and in inlets and straits by the storm tides and lesser variations due to winds and other meteorological disturbances.

496. *Methods employed for current observation and reduction.*—The procedure adopted by the Coast and Geodetic Survey in taking, recording, and reducing current observations is set forth in detail in the Manual of Current Observations (Special Publication No. 215, U. S. Coast and Geodetic Survey). The harmonic reduction and prediction of tides has been explained in chapter II. The harmonic constants of the tides, besides providing the means for tidal predictions, afford an understanding of the variations in the tide, and of the tidal datum planes to which works in tidal waters are referred. Because

of the variation in the currents at different points in a cross section of a tidal waterway, the harmonic constants at individual current stations are not of such general interest, but a summary description of some of the processes employed in their computation may not be out of place.

497. *Harmonic analysis of reversing currents.*—Current measurements at a station in tidal waters usually must be made by a party of some size, from a suitable boat, anchored accurately in position. Consequently current measurements generally are fewer than the tidal observations used for harmonic analysis. If hourly current observations for a 29-day period are available, the harmonic constants for the groups of the principal lunar and solar components, M and S, including their overtides, and the N_2, K, and O_1 components, usually are determined directly from the observations by precisely the process used in determining the tidal harmonic constants. These components are sufficient for current prediction. The overcurrents, such as M_4 and M_6, generally are proportionally larger than the corresponding overtides. If a longer series of observations is available, other components may be included. If hourly observations for a 29-day period are not available, harmonic analyses are made both of the currents at the station during the limited period of the observations, and of the concurrent tides at a standard tide station. The mean value of the amplitude of each current component and of its overcurrents is then found by multiplying the amplitude computed from the observations by the ratio of the established mean amplitude of the corresponding component of the tide at the base station to the tidal amplitude computed from the concurrent observations. The epoch of each current component is found by applying the differences between the initial phases of the current at the current station and the tide at the base station, computed from the concurrent short-term observations, to the established epoch of the tide at the base station, corrected for the difference in the longitudes of the two stations. The predicted hourly heights of the tide at the base station, instead of the recorded heights, usually are preferred for this comparison, since accidental meteorological disturbances of the tide may not produce corresponding changes in the current at another station.

498. *Prediction of currents in tidal straits.*—It was shown in paragraph 256 that if a channel is so short that its currents are but little modified by the storage and release of water in its tidal prism, and if the fluctuating surface head between the entrances produces such strong currents that the flow is essentially frictional, or "hydraulic," the current lags behind the head by but a small angle, and the square of the successive strengths of the current is closely proportional to the nearly concurrent maximum surface heads. The amplitude and phase of each component of the surface head in the strait may be determined,

by the procedure set forth in paragraph 239, from the amplitudes and phases, referred to a common origin of time, of the corresponding components of the tides at stations at the two entrances. The tabulated epoch of each of the tidal components is the difference between the phases of the equilibrium and actual tidal components at the station. To transform these epochs to a common origin of time, they may be converted into Greenwich epochs, by adding the longitude of the station multiplied by the subscript of the component (paragraph 120). After the harmonic constants of the head have been determined, the predicted heights and times of the two daily maximum heads in the strait may be run off on a tide-predicting machine. The relation between the square of the strength of the current at a selected station in the strait and the corresponding head, and the lag of the current with respect to the head, are both determined from the averages of an adequate number of current measurements at the station. From these relations, the predicted times and heights of the heads are readily converted into predicted times and strengths of the current. By applying suitable scales to the tide-predicting machine, the times and strengths of the currents may be read off directly.

499. *Harmonic analysis of rotary currents.*—Data on the rotary offshore currents are provided principally by hourly measurements of the current directions and velocities at the lightships operated by the Lighthouse Service of the United States. The south-north and west-east components of the observed currents are analyzed, and their harmonic constants in each direction determined. It is not difficult to show that the current curve of the resultant of each of the harmonic components, produced by combining its coordinate components in the two directions, is an ellipse. The resultant currents of the component are a maximum and a minimum in the direction of the major and minor axes of the ellipse. The azimuths of these maximum and minimum currents are determined from the coordinate amplitudes and epochs of the component, by a formula whose derivation and application need not be repeated here, and the harmonic constants of the component in these directions determined. The direction of the maximum and minimum velocities of the resultant of all of the components nearly coincides with the axes of the principal lunar semidiurnal component, M_2. By transforming all of the components to these axes, the strengths and times of the current in these directions may be predicted.

500. *Computation of average times of reversing currents.*—The successive times of slack water and of the strengths of the current at a station ordinarily are taken off a plot of the hourly or half-hourly current measurements. The respective intervals after the preceding predicted high or low water at an established reference tidal station in the vicinity, or the intervals after the times of slack and strength at

an established reference current station, are ascertained and averaged. If the tides and currents have much diurnal inequality the intervals of the greater flood and ebb strengths preferably are referred to the times of higher high and lower low water. By adding the established intervals between a lunar transit at Greenwich and the times of the tide or current phases at the reference station, the Greenwich intervals at the given station are then determined. These Greenwich intervals are preferred to lunitidal intervals reckoned from the time of a lunar transit at the station, because the difference between the Greenwich intervals at any two stations gives the difference between the respective times of their currents directly, without any correction for the difference in the longitudes of the stations. If the current had a simple harmonic fluctuation with the speed of the M_2 component, the duration of each increase in velocity from slack to strength, and of each decrease from strength to slack, would be one-quarter of the semilunar day of 12.42 mean solar hours. To establish a single time interval for all four slacks and strengths at a station, the "mean current hour" at the station is computed by averaging the Greenwich intervals of the strength of the flood, the slack before flood increased by 3.10 hours, slack after flood decreased by 3.10 hours, and strength of flood increased or decreased by 6.21 hours, after bringing all of these sums into the same semilunar day by adding or subtracting 12.42 hours as may be necessary.

501. In estuaries and tidal rivers the fresh-water flow may be so great that the current remains in one direction and the velocity varies from a maximum to a minimum without passing through slack. Again, the overcurrents at some stations are so large that the current reaches two maximum velocities during each flood, or ebb, or both. The direction of the current may even reverse between these maxima. The measures taken in these special cases need not be elaborated here.

502. *Reduction of average current strengths.*—Since the *tidal* currents in estuaries and other closed channels, and in inlets to a closed basin, are due to the filling and emptying of the tidal prism of the channel or basin, the successive strengths of the tidal flood and ebb at a current station in the channel are nearly proportional to the concurrent ranges of the tide at a representative tide station on the waterway. The average tidal flood and ebb strengths, determined from a short series of observations, therefore may be converted into long-term averages by multiplying them by the ratio of the established mean tidal range at the tidal station to the average observed range during the period of the current observations. Obviously, this correction is not to be applied to any constant component of the current which may be produced by fresh-water outflow, or other cause, during the period of the observations.

503. Because the strengths of the flood and of the ebb occur at different heights of the tide, the areas of the cross sections of the channel are not the same at both and their velocities would differ somewhat even if the flow were wholly tidal. For the purpose of applying the correction, the tidal parts of flood and ebb strengths are considered to be equal. The tidal current strength at the station during the period of the observations is then taken as one-half of the arithmetic sum of the mean observed flood and ebb strengths, and the nontidal current as one-half of their algebraic sum, with the flood current positive and the ebb negative. These tidal current strengths are corrected to their long-term values by applying the factor derived from the comparative tidal ranges at the reference station. The corrected average flood strength is then derived by adding, algebraically, the nontidal current to the corrected tidal current strength; and the corrected ebb strength by the algebraical subtraction of the nontidal current.

504. At stations in tidal straits, in which the flow is largely frictional and determined almost entirely by the surface head between the entrances, the average tidal current strength derived from a short series of observations is multiplied by the *square root* of the ratio of the established mean range at a suitable tidal station in the waterway to the average observed range during the period of the current observations.

505. *Average polar curves of rotary currents.*—The rotary currents at offshore stations usually are weak and irregular. To prepare an average current curve at a station where the tides and currents are of the semidiurnal type, such as that shown in figure 82, the directions and velocities of all currents observed within half an hour before or after a predicted time of high water at the reference station are summed and averaged to give the average direction and velocity at the time of high water at the reference station; those observed between half an hour and an hour and a half after high water, to give the average direction and velocity 1 hour after high water at the reference station; and so on. The reference times usually extend from 2 hours before to 3 hours after both high and low water at the reference station. Currents of the mixed type, such as those shown in figures 83 and 84, are similarly grouped at the nearest hours at, before and after, higher high, higher low, lower high and lower low water at the reference station.

506. Any average constant current at the station may be determined by resolving either the original observations or their hourly compilations into south-north and west-east components. The algebraic average value of these components in each direction quite evidently is the component of the constant current in that direction. The summation of the component velocities to derive these averages and

the subsequent subtraction of the constant component current, is facilitated by adding to each component velocity an arbitrary constant sufficiently large to make all of the quantities positive. The direction and velocity of the resultant constant current may be obtained from its components, after the subtraction of any arbitrary constant that may have been added for the convenience of computation. The algebraic subtraction of the constant component of the velocity from the hourly current components in either direction, gives the hourly components of the tidal velocity in that direction. The curve of the average tidal velocities proper may then be constructed by finding the resultant hourly tidal currents. If the period of observation is less than a month, the tidal velocities may be reduced to better mean values by multiplying them by the ratio of the established mean tidal range at the reference station to the average range during the period in which the current observations were made.

507. *Wind currents.*—Analyses of the current observations at lightships have afforded useful information on the strength and directions of the currents produced by winds in open waters. The results indicate that as a general rule, along the Atlantic coast, the velocity, in knots, of the current, produced by a wind of some duration, is about 1½ percent of the wind velocity in miles per hour; and along the Pacific coast, about 2 percent. Because of the rotation of the earth, the direction of the current tends to lie to the right of the direction of the wind in the Northern Hemisphere, and to the left in the southern. A Swedish mathematician, V. W. Ekman, has shown that if the depth of the ocean was unlimited, the surface wind currents would have a direction 45° to the right of the wind in the Northern Hemisphere, and 45° to the left in the southern. (Arkiv for Mathematik, Astronomic, 1905). A comparison between the recorded deviation of vessels from their courses and the direction and strength of the winds causing the currents to which the deviations may be attributed, is said to confirm these relative directions of wind and current (Marmer, The Tide, p. 165). Near the coasts, the direction of the current with respect to the wind is modified by the configuration of the coast line. Thus the current observations at the light vessels from San Francisco to Cape Flattery show that the winds from the northeast, southeast, and northwest quadrants produce currents which set 20° to the right of the wind direction, winds from the southwest quadrant produce currents 20° to the left, and winds from the south and west produce currents which set with the wind.

It need not be remarked that these offshore currents are of more concern to the navigator than to the engineer.

APPENDIX I

EQUIVALENTS AND CONSTANTS

EQUIVALENT VELOCITIES

1 knot=1.69 feet per second=1.15 miles per hour.
1 foot per second=0.592 knot=0.682 mile per hour.
1 mile per hour=0.868 knot=1.467 feet per second.

LUNAR TIMES

Mean interval between lunar transits=12.42 mean solar hours.
1 mean lunar hour=1.035 mean solar hours.
1 mean solar hour=0.966 mean lunar hours.

MEAN SPEED, m_2, OF SEMIDIURNAL LUNAR TIDE

In degrees per hour,	28.9841	log 1.46216
In degrees per second,	0.008051	log 7.90586−10
In radians per second,	0.00014052	log 6.14774−10

TABLE XI.—m_2t *in degrees and minutes, for integral values af t from 0 to 69*

t	0	1	2	3	4	5	6	7	8	9
0	0	28°59′	57°58′	86°57′	115°56′	144°55′	173°54′	202°53′	231°52′	260°51′
1	289°50′	318°50′	347°49′	16°48′	45°47′	74°46′	103°45′	132°44′	161°43′	190°42′
2	219°41′	248°40′	277°39′	306°38′	335°37′	4°36′	33°35′	62°34′	91°33′	120°32′
3	149°31′	178°30′	207°29′	236°29′	265°28′	294°27′	323°26′	352°25′	21°24′	50°23′
4	79°22′	108°21′	137°20′	166°19′	195°18′	224°17′	253°16′	282°15′	311°14′	340°13′
5	9°12′	38°11′	67°10′	96°9′	125°8′	154°8′	183°7′	212°6′	241°5′	270°4′
6	299°3′	328°2′	357°1′	26°0′	54°59′	83°58′	112°57′	141°56′	170°55′	199°54′

Acceleration of gravity, g, at sea level, varies from 32.089 feet per second at earth's equator, to 32.234 at poles.

Taking $g=32.16$	log 1.50732
$m_2/g=0.000,00437$	log 4.64042−10
$g/m_2=228,890$	log 5.35958

CIRCULAR CONSTANTS

$\pi=3.14159$	log 0.49715
$2\pi/360=0.01745$	log 8.24188
$8/3\pi=0.8488$	log 9.92882
$3\pi/8=1.1781$	log 0.07118
$\sqrt{3\pi/8}=1.0854$	log 0.03559

Appendix II

REDUCTION FACTORS F(Mn) AND 1.02 F_1 AND CORRECTION FACTOR i/B.

Derivation of F (Mn)

1. As pointed out in paragraph 171, the mean range, Mn, at a tidal station varies slightly with the inclination, I, of the moon's orbit. The factor F(Mn) is applied to reduce a mean range derived from observations extending over a month or year to the true mean value during the 19-year period in which the orbit tilts to and fro as the moon's node makes the circuit of the ecliptic (par. 35). The values of F(Mn) are derived from the relation between the mean range and the harmonic components of the tide at a station.

2. *Relation of high water to the amplitude of the* M_2 *component.*—At most tidal stations the M_2 component is so much larger than the others that high water occurs near the time at which the ordinate of this component is a maximum.

In figure 1, CP is the generating radius of the M_2 component at an instant when its ordinate is a maximum, CR is the radius vector of the resultant of all of the components at that instant, determined by drawing successive lines parallel and equal to the generating radii of the other components (par. 51), and TH is the ordinate of the resultant on the tide curve. At the high water immediately ensuing (or preceding), the radius vector of the resultant is CR', nearly, if not quite, coinciding with the Y axis, its ordinate on the tide curve is $T'H'$, and CP' is the position of the generating radius of the dominant component. Let Δt be the time interval in which R moves to R' and P to P'; v, the corresponding angle between CP and CP'; and let $T'H'—TH=\Delta y$.

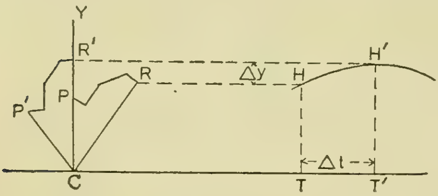

FIGURE 1.

3. The height of mean high water above sea level is the mean of the successive values of TH plus the mean value of Δy. Since the successive values of TH occur at intervals equal to the period of the principal component, and at the instants at which this component is a maximum, their mean is the amplitude, M_2, of this component,

increased or decreased by the constant values of any of its overtides at these instants (par. 78). As the overtides are relatively small, the mean value of TH may be taken, for purposes of computing a correction, as M_2. Δy is always positive, whether high water occurs before or after the high water of the principal component. In the long run, for every value of Δy occurring when high water is in the lead, an equal value will occur when high water lags behind. Neglecting the effect of overtides, the height of mean high water above mean sea level is therefore the amplitude, M_2, of the principal component plus *one-half* of the numerical mean value of Δy.

4. Representing, for generality, the ordinate of the dominant component as $A \cos (at+\alpha)$, and the ordinates of the other components as $B_1 \cos (b_1t+\beta_1)$, $B_2 \cos (b_2t+\beta_2)$, etc., the equation of the tide takes the form:

$$y=A \cos (at+\alpha)+B_1 \cos (b_1t+\beta_1)+B_2 \cos (b_2t+\beta_2)+ \cdots \quad (1A)$$

Since Δy is the change in y due to a relatively small increase, Δt, in t, its value is approximated by differentiating the right-hand member of equation (1), and is:

$$\Delta y=$$
$$-[Aa \sin (at_0+\alpha)+B_1b_1 \sin (b_1t_0+\beta_1)+B_2b_2 \sin (b_2t_0+\beta_2+) \cdots]\Delta t \quad (2A)$$

in which t_0 is a time at which the ordinates of the dominant component is a maximum. Such times occur when $at+\alpha=0$, 2π, 4π, 6π, etc. The value of t_0 is given by the equation:

$$at_0+\alpha=2n\pi$$

whence:

$$t_0=2n\pi/a-\alpha/a \quad (3A)$$

where n is any integer.

Substituting this value in equation (2A):

$$\Delta y=-[Aa \sin 2n\pi+B_1b_1 \sin (2n\pi b_1/a-\alpha b_1/a+\beta_1)$$
$$+B_2b_2 \sin (2n\pi b_2/a-\alpha b_2/a+\beta_2)+ \cdots]\Delta t. \quad (4A)$$

Since the generating radius CP of the dominant component moves through the angle v with the speed a in the time Δt:

$$\Delta t=av$$

Placing for convenience

$$2n\pi b_1/a-\alpha b_1/a+\beta_1=x_1,\ 2n\pi b_2/a-\alpha b_2/a+\beta_2=x_2, \text{ etc.} \quad (5A)$$

Then, since $\sin 2n\pi=0$, equation (4A) reduces to:

$$\Delta y=-[B_1b_1 \sin x_1+B_2b_2 \sin x_2 \cdots]av. \quad (6A)$$

An expression for v remains to be found.

5. The maximum values of y, equation (1A), occur when $dy/dt=0$, or when

$$-Aa \sin (at_1+\alpha)-B_1b_1 \sin (b_1t+\beta_1)-B_2b_2 \sin (b_2t_1+\beta_2)- \cdots =0. \quad (7A)$$

At these maxima, the radius vector, CR', is so close to the Y axis that the angle $R'CP'$ may be taken as equal to $PCP'=v$. Since the generating radius of the dominant component is at CP' at the maximum values of y,

$$at_1+\alpha=2n\pi+v.$$

whence

$$t_1=2n\pi/a-\alpha/a+v/a. \quad (8A)$$

Substituting this value in equation (7A):

$$Aa \sin (2n\pi+v)+B_1b_1 \sin (2n\pi b_1/a-\alpha b_1/a+vb_1/a+\beta_1)$$
$$+B_2b_2 \sin (2n\pi b_2/a-\alpha b_2/a+vb_2/a+\beta_2)+ \cdots =0. \quad (9A)$$

The first term in equation (9A) reduces to $Aa \sin v$. Simplifying the remaining terms by substituting x_1, x_2, etc., for the equivalent expressions given in equation (5A), the equation reduces to:

$$Aa \sin v+B_1b_1 \sin (x_1+b_1v/a)+B_2b_2 \sin (x_2+b_2v/a)+ \cdots =0$$

Expanding the sine functions:

$$Aa \sin v+B_1b_1 \sin x_1 \cos vb_1/a+B_1b_1 \cos x_1 \sin vb_1/a$$
$$+B_2b_2 \sin x_2 \cos vb_2/a+B_2b_2 \cos x_2 \sin vb_2/a+ \cdots =0 \quad (10A)$$

The fractions b_1/a, b_2/a, etc., are the ratios of the speeds of the various components to that of the dominant component. For semidiurnal components these ratios are close to unity, and for diurnal components close to one-half. The angle v is not large at any time unless the tide approaches the diurnal type. The values of $\sin vb_1/a$, $\sin vb_2/a$, etc., are therefore approximately equal to b_1v/a, b_2v/a, etc., respectively, and the values of $\cos vb_1/a$, $\cos vb_2/a$, etc., are nearly unity. Substituting these values, equation (10A) becomes:

$$Aav+B_1b_1 \sin x_1+B_1b_1^2v/a \cos x_1+B_2b_2 \sin x_2+B_2b_2^2v/a \cos x_2+ \cdots =0$$

whence:

$$v=-\frac{B_1b_1 \sin x_1+B_2b_2 \sin x_2+ \cdots}{Aa+B_1b_1^2/a \cos x_1+B_2b_2^2/a \cos x_2+ \cdots} \quad (11A)$$

Substituting this value in equation (6A):

$$\Delta y=\frac{(B_1b_1 \sin x_1+B_2b_2 \sin x_2+ \cdot \cdot \cdot)^2}{Aa^2+B_1b_1^2 \cos x_1+B_2b_2^2 \cos x_2+ \cdot \cdot \cdot}$$

$$=\frac{B_1^2b_1^2 \sin^2 x_1+B_2^2b_2^2 \sin^2 x_2+ \cdot +2B_1B_2b_1b_2 \sin x_1 \sin x_2+ \cdot \cdot \cdot}{Aa^2+B_1b_1^2 \cos x_1+B_2b_2^2 \cos x_2+ \cdot \cdot \cdot} \qquad (12A)$$

6. *Mean value of* Δy.—The symbols x_1, x_2, etc., in equation (12A) represent angles in the form (equation 5A):

$$x=2n\pi b/a-b\alpha/a+\beta$$

where n is an integer.

As successive integral values are assigned to n, x increases by:

$$2\pi b/a=2\pi(b-a)/a+2\pi.$$

At each increase in n, the value of x increases, therefore by $2\pi(b-a)/a$. As the speed, b, of any semidiurnal component does not differ greatly from a, the speed of the dominant component, the fraction $2\pi\ (b-a)/a$ is comparatively small for such components. The successive values of x steadily increase (or decrease) with each increase in n by an angle which describes a small fraction of the circumference. The speeds of the diurnal components (except M_1) differ by a relatively small amount from one-half of that of the dominant component M_2. For these components the value of x steadily increases by a little more or less than 180° with each increase in n. In either case the values of x fall uniformly, in the long run, over the entire range of angles from 0 to 2π, and the mean values of the trigonometric functions of x in equation (12A) become their true mean values as x varies from 0 to 2π. The mean value of sin ^2x between these limits is one-half, while that of cos x, and of the products of the sines of the differently varying angles x_1, x_2, etc., is zero. Aside then from the effects of the M_1 component and the lunar overtides, the mean value of Δy, becomes:

$$\Delta y_0=\tfrac{1}{2}(B_1^2b_1^2+B_2^2b_2^2+ \cdot \cdot \cdot)/Aa^2 \qquad (13A)$$

7. *Mean high water in terms of the harmonic components.*—Since the height of mean high water above mean sea level is the amplitude of the dominant component increased by one-half of the mean value of Δy, it is given by the expression:

$$\begin{aligned}\text{MHW}&=A+\tfrac{1}{4}(B_1^2b_1^2+B_2^2b_2^2+ \cdot \cdot \cdot)/Aa^2\\ &=A[1+\tfrac{1}{4}(B_1^2b_1^2/A^2a^2+B_2^2b_2^2/A^2a^2+ \cdot \cdot \cdot) \qquad (14A)\end{aligned}$$

in which $A = M_2, a = m_2$; B_1, B_2, etc., are the amplitudes of the other harmonic components (except M_1 and the lunar overtides) and b_1, b_2, etc., are the respective speeds of these components.

Since the speeds of the lunar overtides are two, three, and four times the speed of M_2, the successive increments of x in equation (12A) for these components, as n increases by successive integers, are 4π, 6π, and 8π respectively. The successive values of the trigonometric functions of x in that equation are therefore all identical. Similarly the successive increments of x for the M_1 component are each equal π, and for the M_3 component $3/2.\pi$ For all of these components the mean value of $\sin^2 x$ is not ½. The effect of these components on the elevation of mean high water does not therefore follow the law expressed by equation (14A). These components are however generally too small to affect the elevation of mean high water appreciably, and the terms to be added to account for them need not be developed here.

8. *Mean tidal range.*—The elevation of mean low water below mean sea level may be derived in the same manner as the elevation of mean high water above sea level, and with the identical result. The expression for the mean tidal range is therefore:

$$\mathrm{Mn} = 2A[1 + \tfrac{1}{4}(B_1^2 b_1^2/A^2 a^2 + B_2^2 b_2^2/A^2 a^2 + \cdot \cdot \cdot)] \qquad (15\mathrm{A})$$

The factors b_1^2/a^2, b_2^2/a^2, etc., are close to unity for the semidiurnal components, and close to ¼ for the diurnal. The ratios B_1^2/A^2, B_2^2/A^2, are very small for those components whose amplitude is less than one-twentieth of that of the M_2 component. Omitting the components that rarely if ever exceed this ratio, equation (15A) becomes:

$$\mathrm{Mn} = 2\mathrm{M}_2[1 + \tfrac{1}{4}(\mathrm{S}_2^2 s_2^2/\mathrm{M}_2^2 \mathrm{m}_2^2 + \mathrm{N}_2^2 \mathrm{n}_2^2/\mathrm{M}_2^2 \mathrm{m}_2^2 + \mathrm{K}_2^2 \mathrm{k}_2^2/\mathrm{M}_2^2 \mathrm{m}_2^2 + \mathrm{K}_1^2 \mathrm{k}_1^2/\mathrm{M}_2^2 \mathrm{m}_2^2 + \mathrm{O}_1^2 \mathrm{o}_1^2/\mathrm{M}_2^2 \mathrm{m}_2^2 + \mathrm{P}_1^2 \mathrm{p}_1^2/\mathrm{M}_2^2 \mathrm{m}_2^2 + \mathrm{Q}_1^2 \mathrm{q}_1^2/\mathrm{M}_2^2 \mathrm{m}_2^2)] \qquad (16\mathrm{A})$$

9. The numerical value of the mean tidal range derived from equation (16A) is always substantially less than that derived from direct observation. Aside from the effect of overtides and the approximations introduced in the derivation of the formula, this deficiency may be attributed to the fact that any accidental variation in the water elevation occurring near the time of computed high water increases the observed high water by substantially the *maximum* amount of the variation if positive, but decreases the observed high water by but substantially the *minimum* amount of the variation if negative. In the long run, therefore, these variations effect a cumu-

lative increase in the observed high water, and, similarly a cumulative depression of the observed low water. Equation (16A) establishes, however, a logical basis for determining the corrections to be made for the changing inclination of the moon's orbit to the Equator.

10. *Numerical value of* $F(\mathrm{Mn})$.—The amplitudes of the various lunar tidal components during any particular year (or month) are determined by applying the appropriate factor $f=1/F$ to the recorded mean values of these amplitudes (par. 125). For solar components, the value of f is unity. The expression for the mean tidal range during any particular year is then:

$$\begin{aligned}\mathrm{Mn}' = 2f\mathrm{M_2}\{1+\tfrac{1}{4}[(\mathrm{S_2s_2}/f\mathrm{M_2m_2})^2+(f\mathrm{N_2n_2}/f\mathrm{M_2m_2})^2+(f\mathrm{K_2k_2}/f\mathrm{M_2m_2})^2\\ +(f\mathrm{K_1k_1}/f\mathrm{M_2m_2})^2+(f\mathrm{O_1o_1}/f\mathrm{M_2m_2})^2+(\mathrm{P_1p_1}/f\mathrm{M_2m_2})^2\\ +(f\mathrm{Q_1q_1}/f\mathrm{M_2m_2})^2]\}\end{aligned} \tag{17A}$$

The factor to be applied to reduce the mean tidal range, as determined from observations during a particular year, to its true mean value is therefore:

$$F(\mathrm{Mn}) = \mathrm{Mn}/\mathrm{Mn}' \tag{18A}$$

in which the value of Mn is given by equation (16A) and the value of Mn′ is given by equation (17A).

11. The computation of the value of F (Mn) for the true ratios of the amplitudes of the actual components of the tide at a tidal station, and for the successive values of the reduction factors f corresponding to the inclination I of the moon's orbit to the equator, would be a very laborious process, not justified by the accuracy of the results secured. A sufficient approximation is afforded by taking for the ratios of the semidiurnal components the ratios of the mean values of the coefficients of the corresponding equilibrium components, set forth in table IV, paragraph 129. The ratios of the amplitudes of the diurnal components to $\mathrm{M_2}$ vary widely at different tidal stations, but these amplitudes have a fairly consistent ratio between themselves. The index for the amplitude of the diurnal components is therefore taken as the ratio of $\mathrm{K_1+O_1}$ to $\mathrm{M_2}$ at the tidal station, the ratio of the diurnal components to $\mathrm{K_1+O_1}$ being taken as that of the mean values of the coefficients of the corresponding equilibrium components, as given in the same table.

12. Equation (16A) may be written:

$$\begin{aligned}\mathrm{Mn} = 2\mathrm{M_2}\{1+(\mathrm{S_2s_2}/2\mathrm{M_2m_2})^2+(\mathrm{N_2n_2}/2\mathrm{M_2m_2})^2+(\mathrm{K_2k_2}/2\mathrm{M_2m_2})^2\\ +[(\mathrm{K_1+O_1})^2/\mathrm{M_2}^2]\,[(\mathrm{K_1k_1}/2(\mathrm{K_1+O_1})\mathrm{m_2})^2+(\mathrm{O_1o_1}/2(\mathrm{K_1+O_1})\mathrm{m_2})^2\\ +(\mathrm{P_1p_1}/2(\mathrm{K_1+O_1})\mathrm{m_2})^2+(\mathrm{Q_1q_1}/2(\mathrm{K_1+O_1})\mathrm{m_2})^2]\}.\end{aligned}$$

Applying the numerical values of the speeds of the various components and the mean values of the coefficients of the corresponding equilibrium components, this reduces to:

$$\mathrm{Mn}=2\mathrm{M}_2\{1.0717+0.03585(\mathrm{K}_1+\mathrm{O}_1)^2/\mathrm{M}_2^2\}. \qquad (19\mathrm{A})$$

Designating the reduction factors of the several components as $f(\mathrm{M}_2)$, $f(\mathrm{N}_2)$, $f(\mathrm{K}_2)$, etc., and their squares as $f^2(\mathrm{M}_2)$, $f^2(\mathrm{K}_2)$, etc., and noting that $f(\mathrm{M}_2)=f(\mathrm{N}_2)$ and $f(\mathrm{Q}_1)=f(\mathrm{O}_1)$ and that $1/f(\mathrm{M}_2)=F(\mathrm{M}_2)$; equation (17A) similarly reduces, after applying the same numerical values to the amplitudes and speeds of the components, to:

$$\begin{aligned}\mathrm{Mn}'=2\mathrm{M}_2 f(\mathrm{M}_2)\{1.009&+F^2(\mathrm{M}_2)[0.0583+0.0043f^2(\mathrm{K}_2)]\\&+F^2(\mathrm{M}_2)\ ((\mathrm{K}_1+\mathrm{O}_1)/\mathrm{M}_2)^2[0.0025+0.0230f^2(\mathrm{K}_1)\\&+0.0103f^2(\mathrm{O}_1)]\}. \qquad (20\mathrm{A})\end{aligned}$$

Designating for brevity the expressions within the brackets in equations (19A) and (20A) as R and R' respectively:

$$F(\mathrm{Mn})=\mathrm{Mn}/\mathrm{Mn}'=2\mathrm{M}_2R/2\mathrm{M}_2f(M_2)R'=F(\mathrm{M}_2)R/R'. \qquad (21\mathrm{A})$$

The M_2 component may be considered the dominant one when it is not less than $\mathrm{K}_1+\mathrm{O}_1$. The values of $F(\mathrm{Mn})$ for a given value of the inclination of the moon's orbit, I, and of the ratio $(\mathrm{K}_1+\mathrm{O}_1)/\mathrm{M}_2$, when the latter does not exceed unity, may then be found from equation (21A) by substituting in this equation and in the expression for R' the values of $F(\mathrm{M}_2)$, $f(\mathrm{K}_2)$, etc., corresponding to the value of I, as given in the tables contained in manuals on the harmonic analysis of the tides. The determination of the values of $F(\mathrm{Mn})$ for values of $(\mathrm{K}_1+\mathrm{O}_1)/\mathrm{M}_2$ exceeding unity becomes more complicated and need not be here described. The values of $F(\mathrm{Mn})$ are shown in table VI, paragraph 173.

Derivation of $1.02F_1$

13. The factor $1.02F_1$ is applied to the low- and high-water inequalities, DLQ and DHQ, derived from observations during a month or more, to reduce these inequalities, and the consequent elevations of mean lower low and higher high waters, to their astronomical long-term means (par. 189). The diurnal inequalities are due to the diurnal components of the tide at the station. Since the equilibrium components have the same relation to their long-term means as the actual components, the expression for the reduction factor may be derived from the diurnal equilibrium components. For this purpose only the K_1, O_1, and P_1 components need be considered;

since, as shown in table IV, paragraph 129, the amplitudes of the other diurnal equilibrium components are relatively small.

14. The resultant of the diurnal components may be termed the diurnal wave, and its varying amplitude designated D_1. The diurnal wave increases one of the two daily high waters of tides of the semi-diurnal and mixed types, and decreases the other. Since the diurnal wave keeps in general step with the semidiurnal tidal fluctuations, the consequent diurnal inequalities during any period is taken as proportional to the mean value of D_1 during that period.

15. *Long-term mean value of D_1.*—As K_1 is the largest diurnal equilibrium component, the approximate long-term mean value of D_1 is, from equation (14A):

$$Dm = K_1[1 + (O_1 o_1/2K_1 k_1)^2 + (P_1 p_1/2K_1 k_1)^2].$$

The corresponding long-term mean value of the resultant of the K_1 and O_1 components only is:

$$Rm = K_1[1 + (O_1 o_1/2K_1 k_1)^2].$$

Whence:

$$Dm/Rm = 1 + (P_1 p_1/2K_1 k_1)^2/[1 + (O_1 o_1/2K_1 k_1)^2].$$

By substituting the speeds and the mean values of the coefficients of the components, the long-term mean value of D_1 is found to be approximately 1.02 times the mean value of the resultant of the K_1 and O_1 components only.

The amplitude of the resultant of the K_1 and O_1 components fluctuates between $K_1 + O_1$, and $K_1 - O_1$ during the period of one-half a tropical month. Its mean value may be written:

$$Rm = C(K_1 + O_1).$$

In which C is a constant which need not here be determined.

The long-term mean value of D_1 is then:

$$Dm = 1.02C(K_1 + O_1). \tag{22A}$$

16. *Monthly mean value of D_1.*—During a month in which the moon's declination is I, the amplitudes of the K_1 and O_1 equilibrium components are $K_1 f(K_1)$ and $O_1 f(O_1)$ respectively, $f(K_1)$ and $f(O_1)$ being the reduction factors for this value of I. Since P_1 is a solar component, its amplitude remains constant. It combines with the K_1 component into a resultant whose amplitude fluctuates between a maximum and a minimum in a period of the half tropical year. The angle between this resultant and the K_1 component changes but

little in a half tropical month. Designating the length of this resultant at the middle of the month as K', the mean value of D_1 during the month is, very nearly:

$$Dm' = C(K' + O_1 f(O)) \tag{23A}$$

in which C has the same numerical value as in equation (22A).

The correction to be added to the value of K_1 for the month, to give the value of K', is to be derived.

17. *Correction for* P_1.—As shown in paragraph 122 the equation of the K_1 equilibrium component is:

$$y_1 = K_1 \cos (T + h - 90° - \nu')$$

and, from equation (69) that of the P_1 component is

$$y_2 = P_1 \cos (T - h + 90°).$$

The angle between them is:

$$\beta' = 2h - 180° - \nu'.$$

in which h is the mean longitude of the sun (par. 105). Its value on any given day of the year is substantially the same from year to year. It increases at the rate of 0.041° per solar hour, or about 1° per day. ν' is a small angle, which varies with N, the longitude of the moon's node. Its values corresponding to values of N are tabulated in manuals on the harmonic analysis of tides. The value of β' on any date may be corrected for ν' by taking the value of h on half as many days before the given date as there are degrees in ν' when ν' is positive and after the given date when ν' is negative. When so corrected the value of β' is

$$\beta' = 2h - 180°$$

The length, K', of the resultant of the K_1 and P_1 components is easily shown to be

$$\begin{aligned} K' &= \sqrt{K_1^2 + P_1^2 - 2K_1P_1 \cos \beta'} \\ &= \sqrt{K_1^2 + P_1^2 - 2K_1P_1 \cos 2h} \end{aligned}$$

Placing $K' = cK_1$

$$c = K'/K_1 = \sqrt{1 + P_1^2/K_1^2 - 2(P_1/K_1) \cos 2h} \tag{24A}$$

Taking the value of P_1/K_1 as the ratio of the mean values of the coefficients of the corresponding equilibrium components, or as $0.0880/0.2655 = 0.3315$, equation (22A) becomes:

$$c = \sqrt{1.11 - 0.663 \cos 2h} \tag{25A}$$

The value of c on any day of the year may be computed from equation (25A) by substituting the value of h on that day. At the vernal and autumnal equinoxes, March 22 and September 21, $h=0$ and $180°$ respectively, and c has a minimum value of 0.668. At the summer and winter solstices, June 22 and December 22, $h=90°$ and $270°$, and c has a maximum value of 1.331.

The correction to be added to the value of K_1 for the month, to give the value of K', is then

$$K'-\mathrm{K}_1=c\mathrm{K}_1-\mathrm{K}_1=(c-1)\mathrm{K}_1 \tag{26A}$$

18. *Expression for 1.02 F_1.*—Substituting in equation (23A) the expression for K' given in equation (26A)

$$Dm'=C[\mathrm{K}_1 f(\mathrm{K}_1)+(c-1)\mathrm{K}_1+\mathrm{O}_1 f(\mathrm{O}_1)]$$

and the reduction factor is:

$$\begin{aligned} Dm/Dm' &= 1.02C(\mathrm{K}_1+\mathrm{O}_1)/C[((c-1+f(\mathrm{K}_1))\mathrm{K}_1+\mathrm{O}_1 f(\mathrm{O}_1)] \\ &= 1.02(1+\mathrm{K}_1/\mathrm{O}_1)/[((c-1+f(\mathrm{K}_1))\mathrm{K}_1/\mathrm{O}_1+f(\mathrm{O}_1)] \end{aligned}$$

The ratio K/O_1 of the mean values of the equilibrium components is taken as 1.4066.

The reduction factor is written:

$$1.02F_1$$

in which:

$$F_1=2.4066/[1.4066(c-1+f(\mathrm{K}_1))+f(\mathrm{O}_1)]$$

By substituting the values of c, $f(\mathrm{K}_1)$ and $f(\mathrm{O}_1)$ at the middle of each month, the values of 1.02 F_1 may be found as shown in table VIII, paragraph 189.

Approximate value of $(\mathrm{K}_1+\mathrm{O}_1)/\mathrm{M}_2$

19. The statement was made in paragraph 175, that in the lack of better information the ratio $(\mathrm{K}_1+\mathrm{O}_1)/\mathrm{M}_2$ for entering table VI is taken as 2 (DHQ+DLQ)/Mn. It is not difficult to see that the daily high water inequality, DHQ, closely approximates $D_1 \cos \alpha$, where D_1 is the length of the resultant of the diurnal components and α is the angle between the position of its radius vector at high water and the Y axis. At the next low water the radius vector of the resultant of the semidiurnal components has moved through approximately 180°, and that of the diurnal components through approximately 90°. The daily low water inequality is therefore about equal to $D_1 \sin \alpha$, and the sum of the two daily inequalities to

$$D_1 (\cos \alpha+\sin \alpha)$$

As cos α and sin α are both essentially positive, the factor

$$(\cos \alpha + \sin \alpha)$$

has values lying between the comparatively restricted range of a minimum of unity, when α is 0 or 90°, and a maximum of 1.414 when $\alpha = 45°$. The value of DHQ+DLQ is the mean value of

$$D_1 (\cos \alpha + \sin \alpha).$$

The value of C, in equation (22A) of this appendix may be shown to be approximately 0.66. The mean value of DHQ+DLQ is then close to but generally less than K_1+O_1. The value of Mn is similarly close to, but a little more than $2M_2$. It follows therefore, that very roughly:

$$(\mathrm{DHQ}+\mathrm{DLQ})/\mathrm{Mn} = (K_1+O_1)/2M_2$$

$$2(\mathrm{DHQ}+\mathrm{DLQ})/\mathrm{Mn} = (K_1+O_1)/M_2$$

The ratio 2 (DHQ+DLQ)/Mn generally is somewhat less than that of $(K_1+O_1)/M_2$, but the values of F(Mn) in table VI change so slowly with this ratio that no large error is introduced by using this approximation in entering the table.

Correction factor i/B

20. As stated in paragraph 261, chapter V, the correction to the primary current therein designated as i is such that the corrected velocity:

$$B \sin (at+\beta) + i = B[\sin (at+\beta) + i/B]$$

satisfies the general equation of motion (equation 112) when the surface slope has the simple harmonic fluctuation, $S \cos (at+H°)$, and the velocity head term is dropped. Placing, for convenience, $i/B = z$, equation (112) therefore becomes:

$$S \cos (at+H°) + (1/g)\partial B[\sin (at+\beta)+z]/\partial t \pm B^2[\sin (at+\beta)+z]^2/C^2r = 0$$

or:

$$S \cos (at+H°) + (aB/g) \cos (at+\beta) + (B/g)\partial z/\partial t$$
$$\pm B^2[\sin (at+\beta)+z]^2/C^2r = 0. \qquad (27A)$$

The values of B and β are such that equation (145), paragraph 243,

$$S \cos (at+H°) + (aB/g) \cos (at+\beta) + (8/3\pi)(B^2/C^2r) \sin (at+\beta) = 0$$

is identically true for all values of t. Equation (27A) therefore may be written:

$$(B/g)\partial z/\partial t - (8/3\pi)(B^2/C^2r) \sin (at+\beta) \pm (B^2/C^2r)[\sin (at+\beta)+z]^2 = 0.$$

Dividing by B^2/C^2r;

$$(C^2r/Bg)\partial z/\partial t-(8/3\pi)\sin(at+\beta)\pm[\sin(at+\beta)+z]^2=0.$$

From equation (153), paragraph 244:

$$C^2r/Bg=(8/3\pi)\tan\phi/a.$$

Giving:

$$(8/3\pi)\tan\phi\partial z/\partial at-(8/3\pi)\sin(at+\beta)\pm[\sin(at+\beta)+z]^2=0. \quad (28A)$$

Expanding the last term of equation (28A), the differential equation for z becomes:

$$(8/3\pi)\tan\phi\partial z/\partial at-(8/3\pi)\sin(at+\beta)\pm\sin^2(at+\beta)$$
$$\pm 2z\sin(at+\beta)\pm z^2=0. \quad (29A)$$

The correction factor, $z=i/B$, is therefore a function of the angular lag, ϕ, and the phase, $at+\beta$, of the primary current.

21. Since z is relatively small, a first approximation to its value for given values of ϕ and $at+\beta$ may be derived by dropping its square from equation (29A) and neglecting its effect upon the sign of the velocity.

Rearranging, equation (29A) then becomes:

$$(8/3\pi)\tan\phi\partial z/\partial at+2z[\pm\sin(at+\beta)]$$
$$-(8/3\pi)\sin(at+\beta)\pm\sin^2(at+\beta)=0 \quad (30A)$$

in which the positive sign is to be applied when $\sin(at+\beta)$ is positive and the negative sign when it is negative. Angles are in radians.

Equation (30A) does not appear integrable, but the values of z for a given value of ϕ, and for successive values of $at+\beta$ increasing by sufficiently small increments may be derived by a somewhat laborious arithmetical solution. The increment selected, in degrees, will be designated Δat°. Its value in radians is then $\pi\Delta at^\circ/180$. Since differential equations remain approximately true when small finite increments are substituted for the differentials, the first term may be written:

$$(8/3\pi)\tan\phi\Delta z/(\pi\Delta at^\circ/180)$$

in which Δz is the increase in z due to an increment of Δat° degrees in the phase of the primary current. If Δat° is sufficiently small, the value of Δz does not differ materially from the increase in z during the preceding increment in the phase. Designating the preceding value of z as z_0, the first term of equation (30A) then becomes:

$$(480/\pi^2\Delta at^\circ)\tan\phi(z-z_0)=b(z-z_0)$$

in which the coefficient $b=(480/\pi^2\Delta at^\circ)\tan\phi$ may be computed from the given value of ϕ and the selected increment Δat°.

In the second term of equation (30A) the negative sign is prefixed to $\sin(at+\beta)$ when this function is negative. The factor $\pm\sin(at+\beta)$ is then positive for all values of at, and will be so distinguished by writing it as $\overline{\sin(at+\beta)}$. The algebraic sum of the last two terms, for values of at at the selected intervals, may be designated as $-R$. Equation (30A) then becomes:

$$b(z-z_0)+2z\,\overline{\sin(at+\beta)}-R=0$$

whence

$$z=(z_0-R/b)/[1+2\,\overline{\sin(at+\beta)}/b]. \tag{31A}$$

22. The values of z for successive values of $at+\beta$ may be computed from equation (31A) after an initial determination of z_0 has been made. By taking $\Delta\,at^\circ$ as an integral factor of 180°, these values are repeated after $at+\beta$ has passed through 360°. Taking then z_0 as zero at any value of $at+\beta$, such as zero, the resulting values of z may be successively computed through 360°, a corrected initial value of z_0 derived, and the procedure repeated. Since the divisor of the second term of equation (31A) is greater than unity, the new values of z successively approach and finally coincide with those previously found. The process is in fact abbreviated, since the values of z repeat themselves, with the sign reversed, after passing 180°.

23. *Second correction.*—When the flow is largely frictional, and ϕ consequently is a relatively small angle, the values of z derived from the foregoing procedure are so large that their squares are not negligible, and are sufficient, also, to reverse the sign of the velocity when the primary current is small. A further correction, δ, is therefore required. Designating the first determination of the correction factor as z_1, the corrected current becomes $B[\sin(at+\beta)+z_1+\delta]$.

Equation (112) then takes the form:

$$S\cos(at+H^\circ)+(aB/g)\cos(at+\beta)+(B/g)(\partial z_1/\partial t+\partial\delta/\partial t)$$
$$\pm B^2[\sin(at+\beta)+z_1+\delta]^2/C^2r=0$$

which, by a procedure paralleling that in paragraph 20, may be transformed into:

$$(8/3\pi)\tan\phi(\partial\delta/a\partial t+\partial z_1/a\partial t)-(8/3\pi)\sin(at+\beta)$$
$$\pm[\sin(at+\beta)+z_1]^2\pm2\delta[\sin(at+\beta)+z_1]\pm\delta^2=0$$

The approximate values of δ may be derived by dropping its square and neglecting its effect on the sign of the velocity. The errors introduced by these approximations are, it may be observed, much less than those resulting from the same approximations in deriving the initial values of z. The resulting equation may be written:

$$\begin{aligned}&(8/3\pi)\tan\phi\partial\delta/a\partial t+2\delta\{\pm[\sin(at+\beta)+z_1]\}\\&+(8/3\pi)\tan\phi\partial z_1/a\partial t-(8/3\pi)\sin(at+\beta)\\&\pm[\sin(at+\beta)+z_1]^2=0\end{aligned}\tag{32A}$$

in which the positive sign is to be used when the primary current, corrected by z_1, is positive, and negative when it is negative.

By using the same increment, $\Delta at°$, as in the first determination of z, equation (32A) becomes:

$$b(\delta-\delta_0)+2\delta[\overline{\sin(at+\beta)+z_1}]-R=0\tag{33A}$$

in which b has the value previously determined, $\overline{\sin(at+\beta)+z_1}$ is the numerical value of the velocity as first corrected and:

$$-R=(8/3\pi)\tan\phi\partial z_1/\partial at-(8/3\pi)\sin(at+\beta)\pm[\sin(at+\beta)+z_1]^2\tag{34A}$$

The first term in this expression for R may be evaluated by placing

$$(8/3\pi)\tan\phi\partial z_1/\partial at=(8/3\pi)\tan\phi\Delta z_1/(\pi\Delta at°/180)=b\Delta z_1$$

in which Δz_1 is the average of the increments of z_1 for the preceding and ensuing increments of $at+\beta$.

It may be observed that R is the residual by which the first member of equation (28A) differs from zero when the first approximation to a value of z is substituted therein.

From equation (33A):

$$\delta=(\delta_0-R/b)/[1+2[\overline{\sin(at+\beta)+z_1}]/b]\tag{35A}$$

The values of δ for successive values of $at+\beta$ may be computed from equation (35A) by the same process as that employed in computing z from equation (31A).

A second correction may be applied, by the same procedure, if the corrected values of $z_2=z_1+\delta$ give residuals of more than negligible magnitude when substituted in equation (34A).

24. The increments $\Delta at°$ used in computing the correction factors shown in table X, paragraph 261, ranged from 2½°, for small values of ϕ, up to 10° for the small values of i/B when $\phi=80°$.

When $\phi=0$, equation (28A) reduces to:

$$-(8/3\pi)\sin(at+\beta)\pm[\sin(at+\beta)+z]^2=0$$

Whence

$$z=\sqrt{\pm(8/3\pi)\sin(at+\beta)}-\sin(at+\beta) \tag{36A}$$

As the positive sign is applied when $(at+\beta)$ is positive, and the negative sign when it is negative, all of these values are real.

The limiting values of $i/B=z$, for $\phi=0$, shown in table X, are derived from equation (36A).

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